parametric spline curves

curves

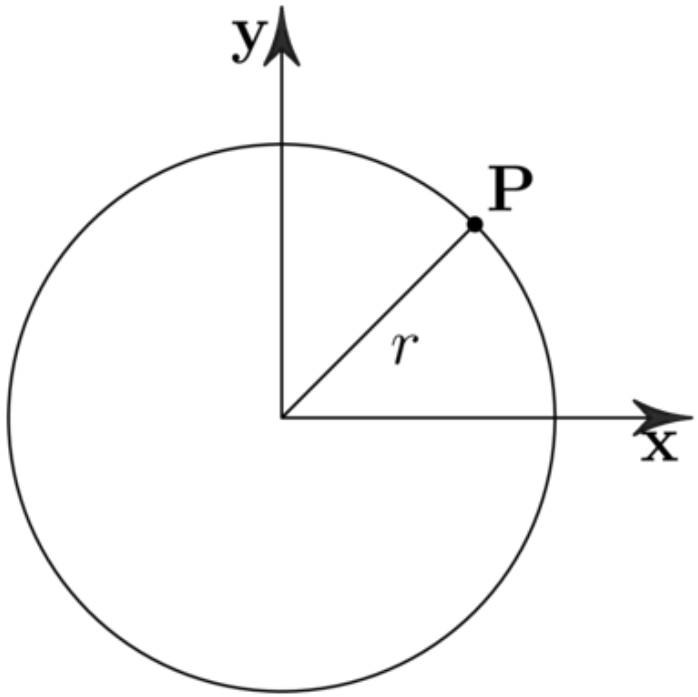
- used in many contexts
  - fonts (2D)
  - animation paths (3D)
  - shape modeling (3D)
- different representation
  - implicit curves
  - parametric curves (mostly used)
- 2D and 3D curves are mostly the same
  - use vector notation to simplify treatment

implicit representation

- implicit curve representation

$$f(\mathbf{P}) = 0$$

- e.g. XY circle: $f(\mathbf{P}) = 0 \rightarrow x^2 + y^2 - r^2 = 0$

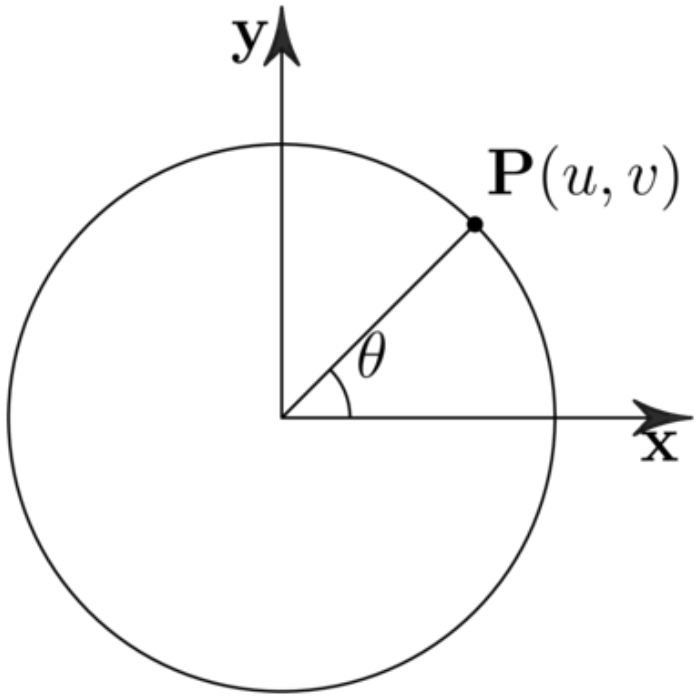


parametric representation

- parametric curve representation

$$\mathbf{P}(u) = (f_x(u), f_y(u), f_z(u))$$

- e.g. XY circle: $\mathbf{P}(u) = (r\cos\theta, r\sin\theta, 0)$



parametric representation

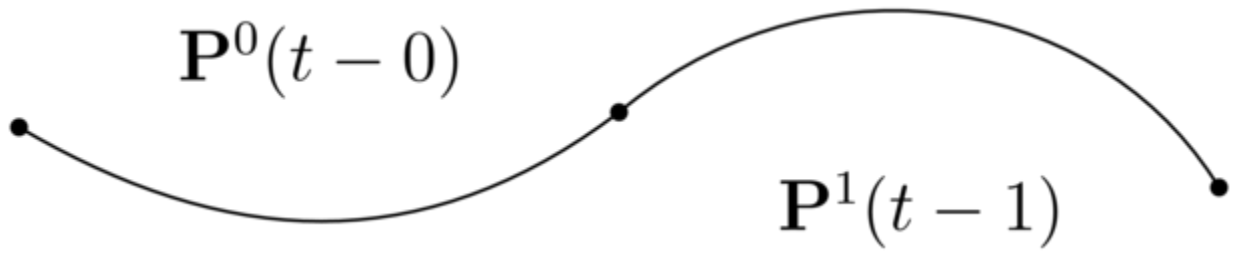
- goals when defining f
  - smoothness
  - efficiency
  - local control
    - curve controls only affect a piece of the curve
  - predictable behaviour
    - easy to use for designers

splines

splines

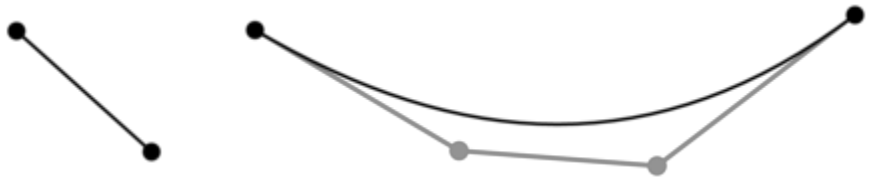
- splines: piecewise parametric polynomials
  - split curve $\mathbf{P}(t)$ into N segments $\mathbf{P}^s(t - t^s)$ at intervals t^s
  - uniform splines: split at integer intervals $t_s \in 0, 1, 2 \cdots$
  - each segment $\mathbf{P}^s$ is a polynomial: smooth and efficient
  - piecewise curve, i.e. splitting: local control

$$\mathbf{P}(t) = \mathbf{P}^s(t - t^s) \text{ with } t \in [0, N), t - t^s \in [0, 1)$$

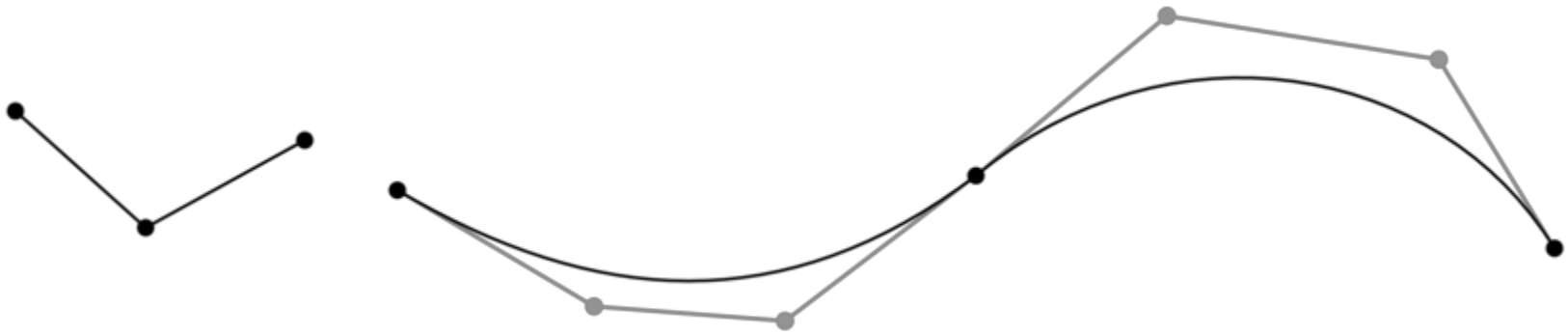


splines

- intuition: define segment by "blending" control points

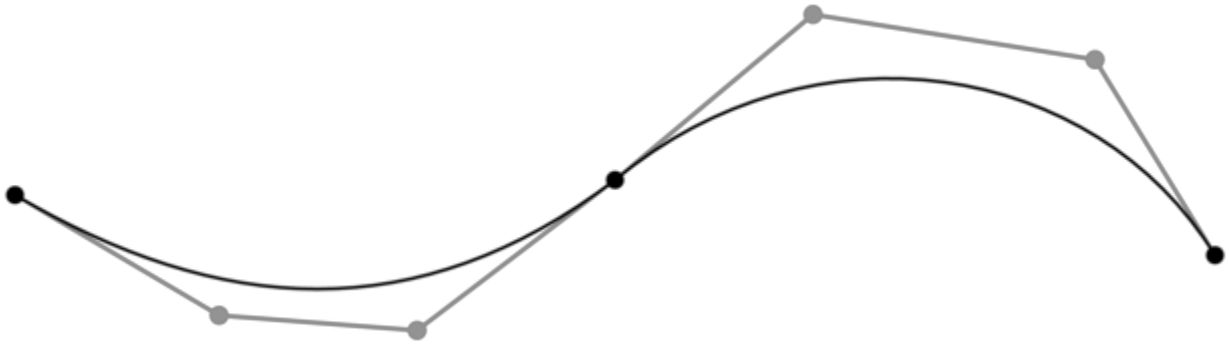


- intuition: join segments to from a curve

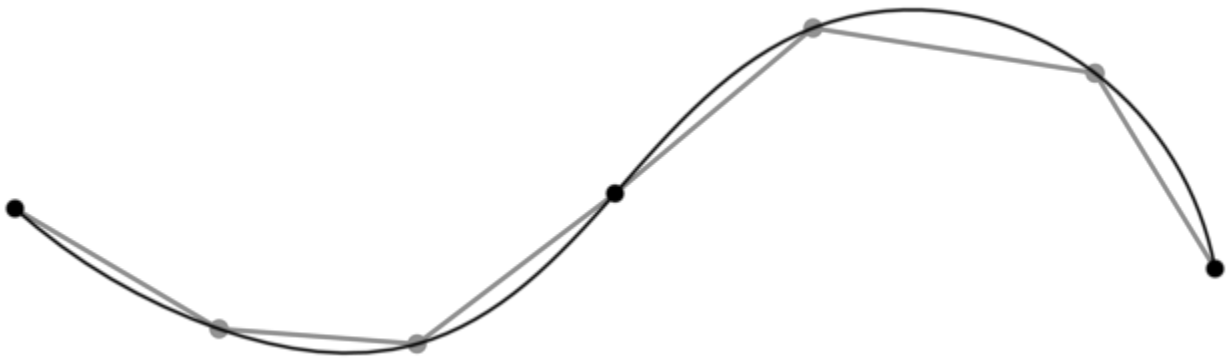


splines

- approximating: guided by control points



- interpolating: pass through control points

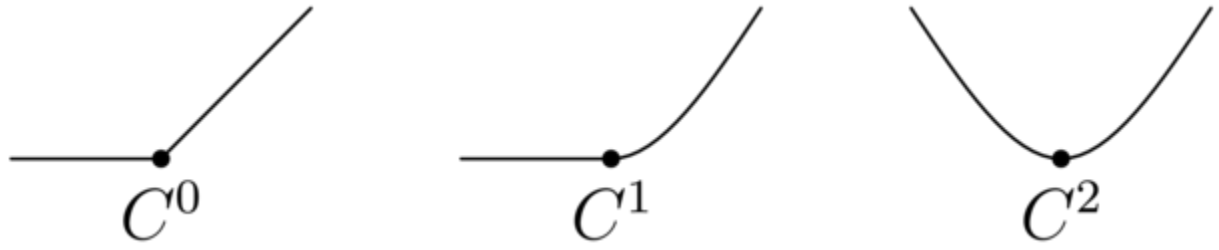


defining splines

- pick segment interpolating function
  - smoothness
- pick segment control points
  - local control
- impose constraints to define segments
  - join segments together
  - ensure smoothness

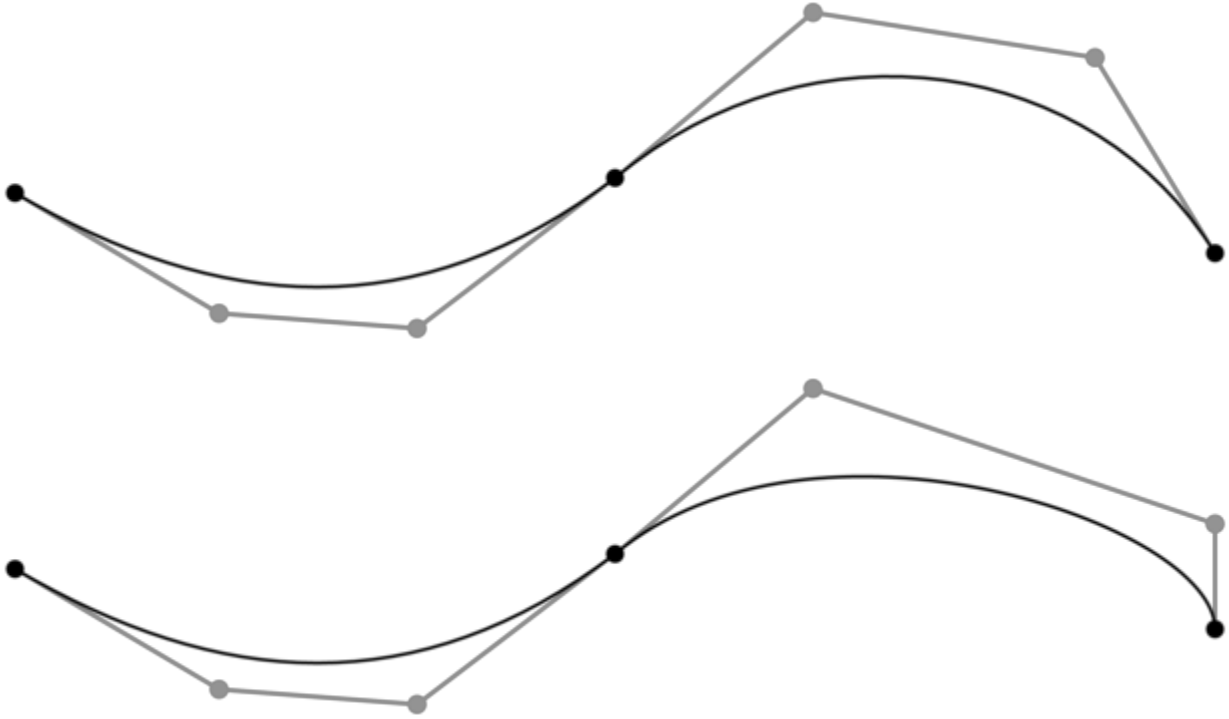
defining splines

- smoothness described by degree of continuity
  - C^0: same position at each side of joints
  - C^1: same tangent at each side of joints
  - C^2: same curvature at each side of joints
  - C^n: n-th derivative defined at joints



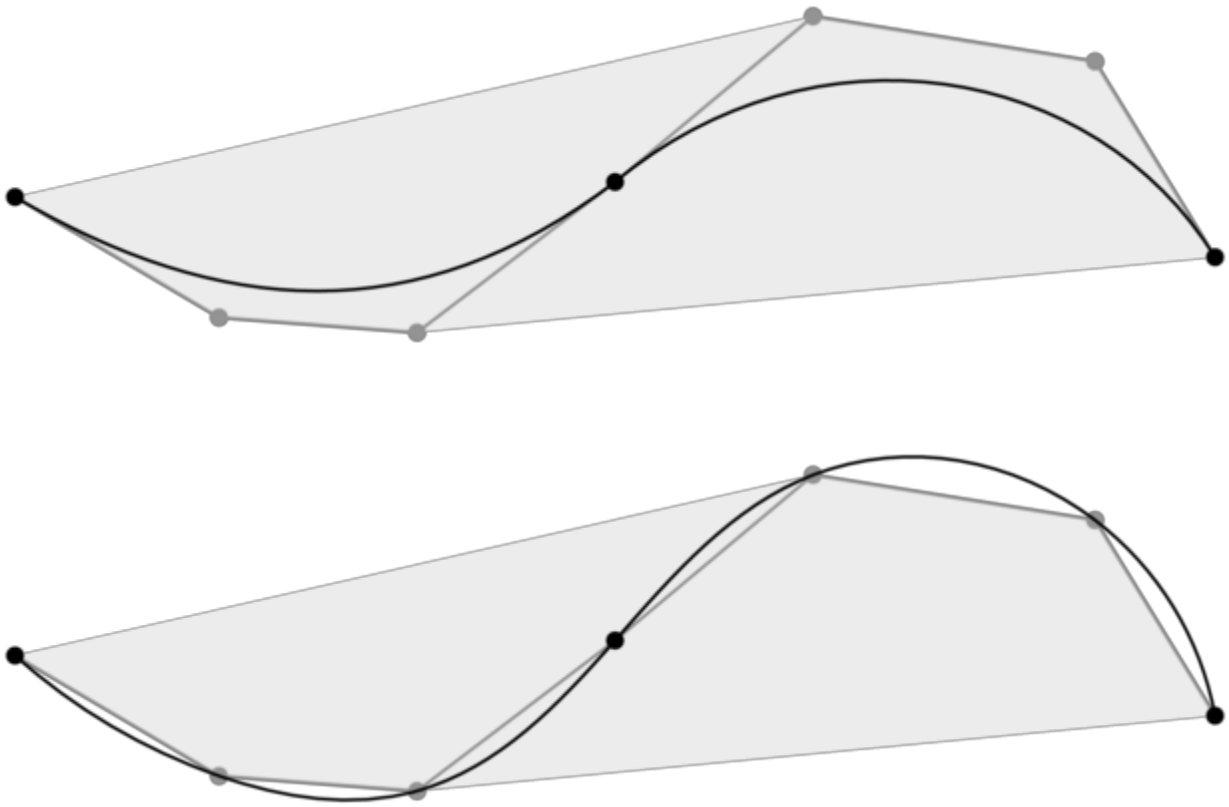
defining splines

- local control: control points affects the curve locally
  - easy to control, true for all splines



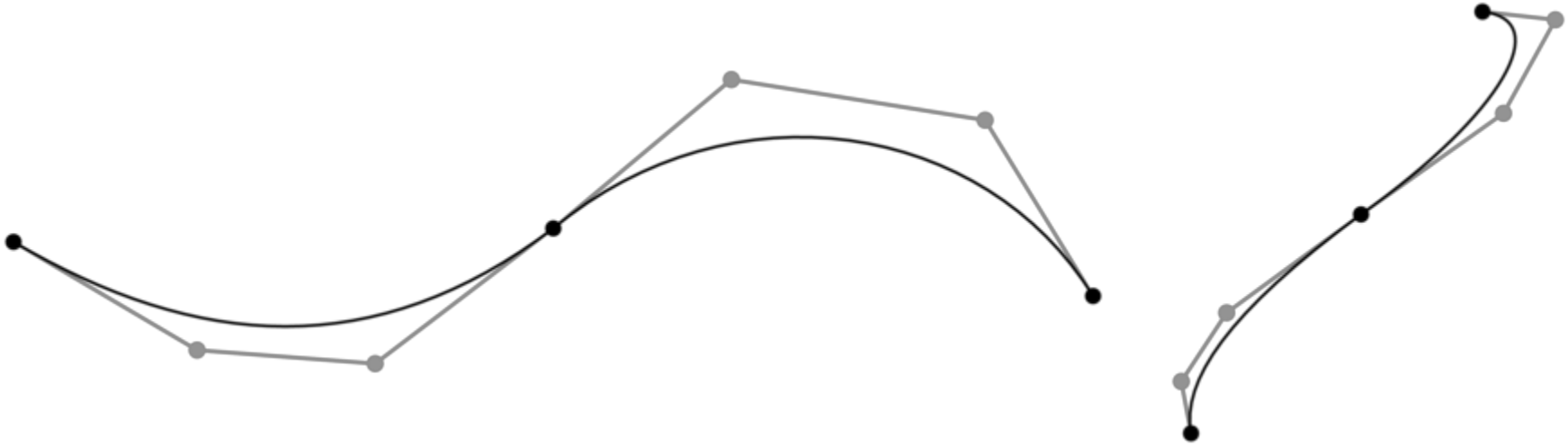
defining splines

- convex hull: smallest convex region enclosing all points
- convex hull property: curve lies in control points convex hull
  - predictable and efficient, but only some splines



defining splines

- affine invariance: transform controls equiv. transform spline
  - efficient, all splines



linear splines

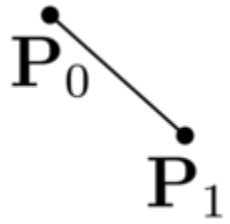
linear splines

- segment: linear function

$$\mathbf{P}(t) = t\mathbf{a} + \mathbf{b} \text{ with } t \in [0, 1)$$

- control points: end points $\mathbf{P}(0) = \mathbf{P}_0$ and $\mathbf{P}(1) = \mathbf{P}_1$

$$\mathbf{P}(t) = (1-t)\mathbf{P}_0 + t\mathbf{P}_1$$

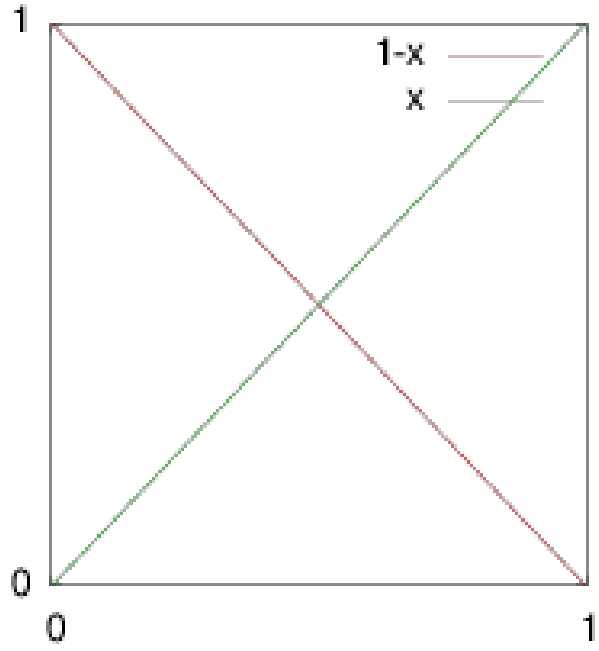


linear splines

- blending functions: interpret as blending control points

$$\mathbf{P}(t) = b_0(t)\mathbf{P}_0 + b_1(t)\mathbf{P}_1$$

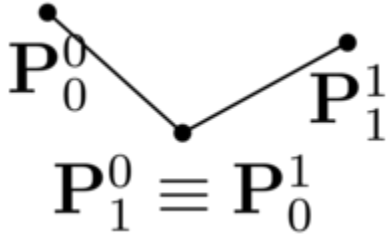
$$b_0(t) = (1-t)$$
$$b_1(t) = t$$



linear splines

- joining segments: impose C^0 continuity
  - segments share endpoints
- C^1 continuity only for straight lines

$$\mathbf{P}^0(1) = \mathbf{P}^1(0) \rightarrow \mathbf{P}_1^0 = \mathbf{P}_0^1$$



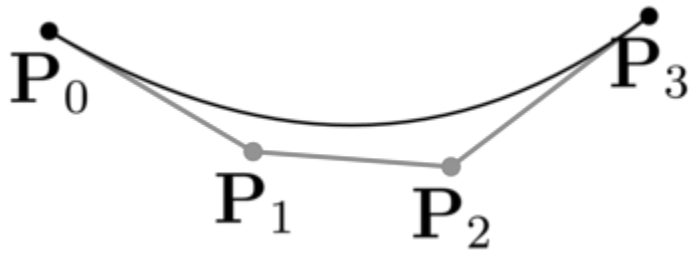
bezier cubic splines

bezier cubic splines

- segment: cubic function

$$\mathbf{P}(t) = t^3\mathbf{a} + t^2\mathbf{b} + t\mathbf{c} + \mathbf{d} \text{ with } t \in [0, 1)$$

- control points: end points $\mathbf{P}_0, \mathbf{P}_3$ and tangents $\mathbf{P}'(0) \propto \mathbf{P}_1 - \mathbf{P}_0$ and $\mathbf{P}'(1) \propto \mathbf{P}_2 - \mathbf{P}_3$

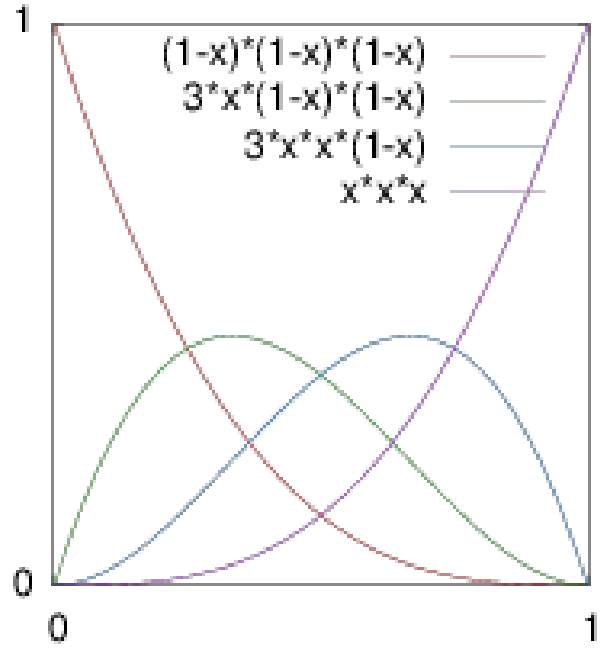


bezier cubic splines

- blending functions: Bernstein polynomials

$$\mathbf{P}(t) = b_0(t)\mathbf{P}_0 + b_1(t)\mathbf{P}_1 + b_2(t)\mathbf{P}_2 + b_3(t)\mathbf{P}_3$$

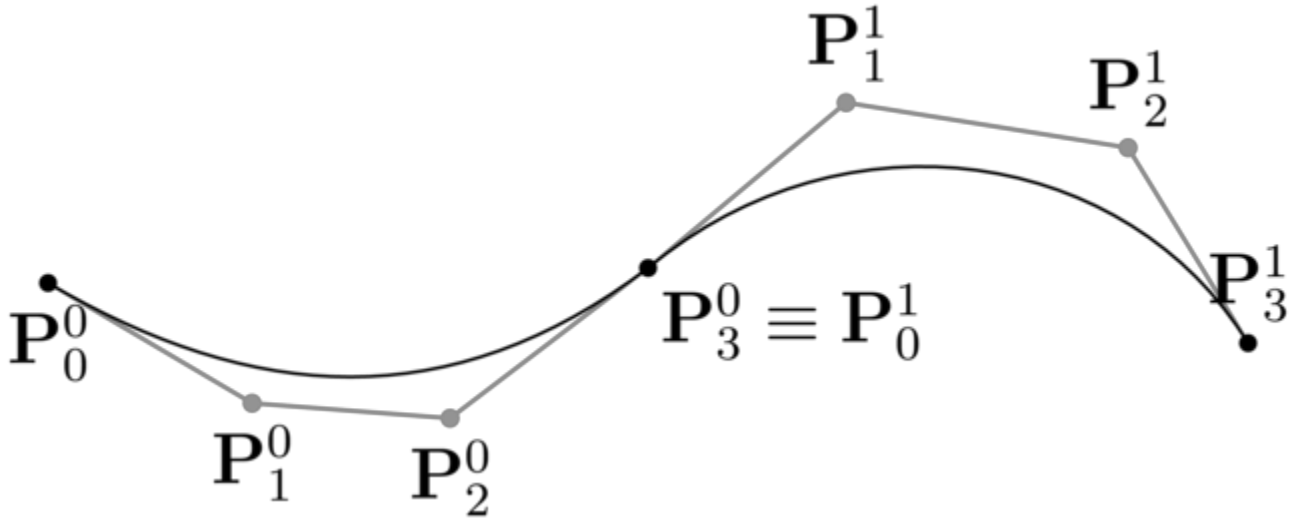
$$b_0(t) = (1-t)^3$$
$$b_1(t) = 3t(1-t)^2$$
$$b_2(t) = 3t^2(1-t)$$
$$b_3(t) = t^3$$



bezier cubic splines

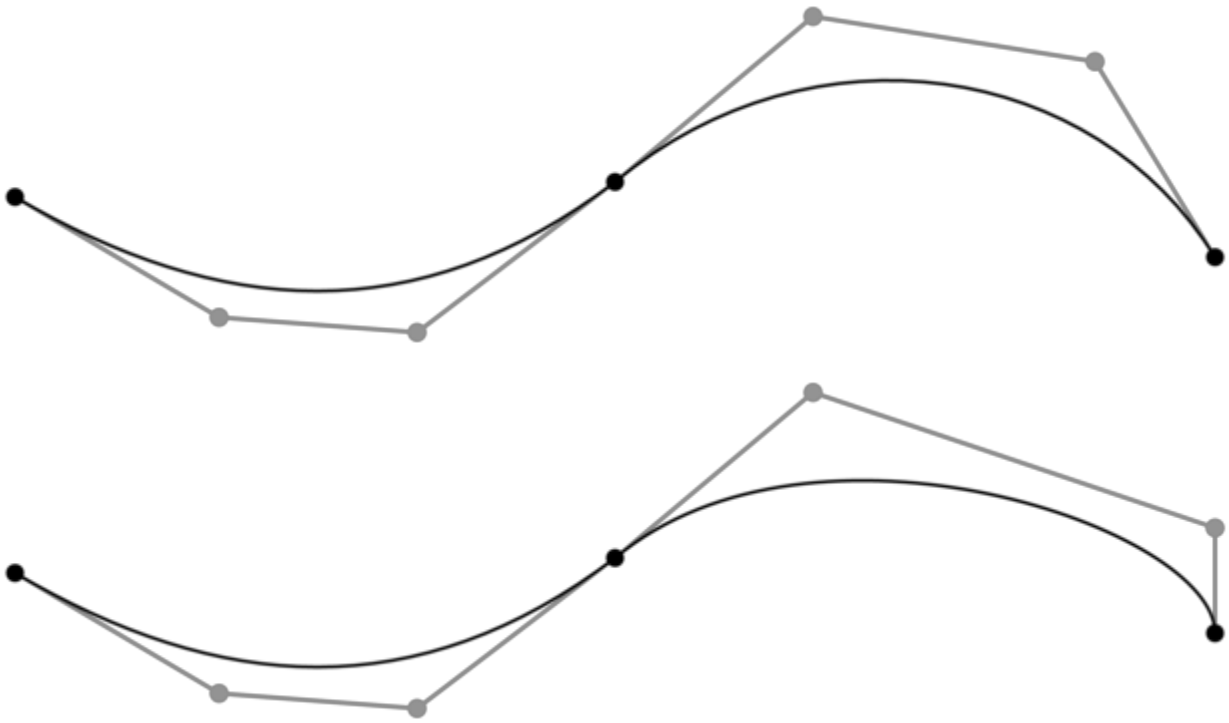
- joining segments: impose C^0 continuity
  - segments share endpoints
- C^1 continuity by collinear tangents

$$\mathbf{P}^0(1) = \mathbf{P}^1(0) \rightarrow \mathbf{P}_1^0 = \mathbf{P}_0^1$$



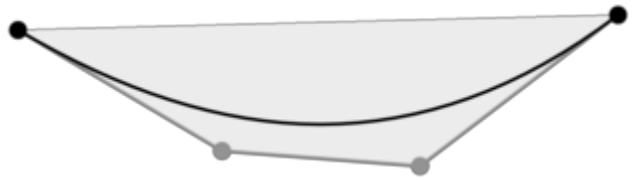
bezier cubic splines

- properties: local control
  - comes from the formulation by segments
  - for each segment, curve defined by 4 control points



bezier cubic splines

- properties: convex hull
  - each segment is convex sum of control points
  - since $b_i(t) \geq 0$ and $\sum_i b_i(t) = 1$



bezier cubic splines

- properties: affine invariance

$$
\begin{aligned}
X(\mathbf{P}(t)) &= M\mathbf{P}(t) + \mathbf{t} = \\
&= M\left(\sum_i b_i(t)\mathbf{P}_i\right) + \mathbf{t} = \\
&= \sum_i b_i(t) M\mathbf{P}_i + \left(\sum_i b_i(t)\right)\mathbf{t} = \\
&= \sum_i b_i(t)(M\mathbf{P}_i + \mathbf{t}) = \sum_i b_i(t) X(\mathbf{P}_i)
\end{aligned}
$$

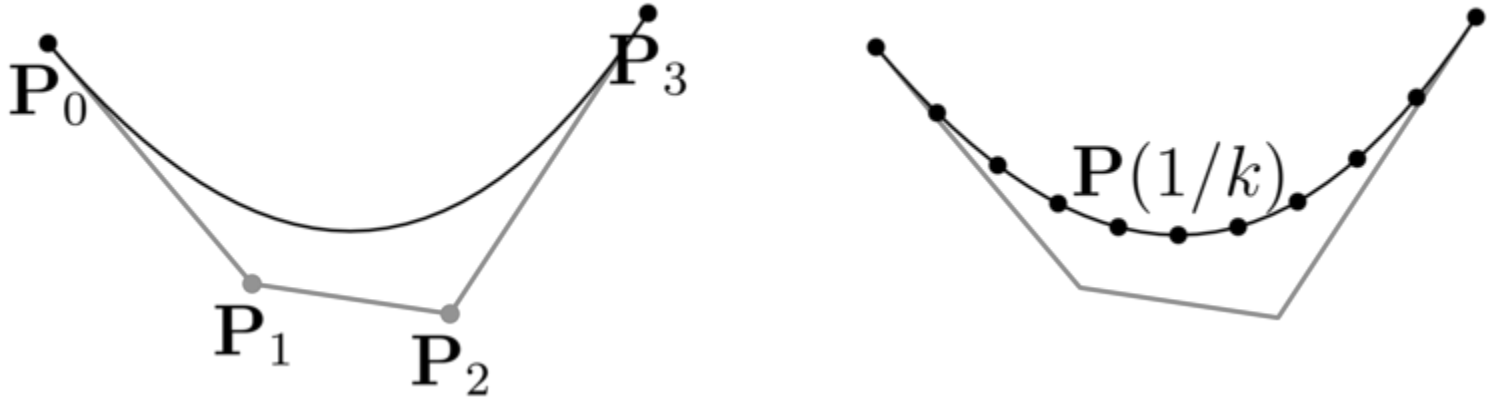
rendering splines

rendering splines

- tesselation: approximate splines with line segments
  - segments are efficient to draw in hardware and software
  - more segments to provide better approximation
- uniform tesselation: split t interval uniformly
  - fast to compute and simple to implement
  - generates many segments
- adaptive tesselation: split recursively until good enough
  - more complex to implement
  - less segments with guaranteed approximation

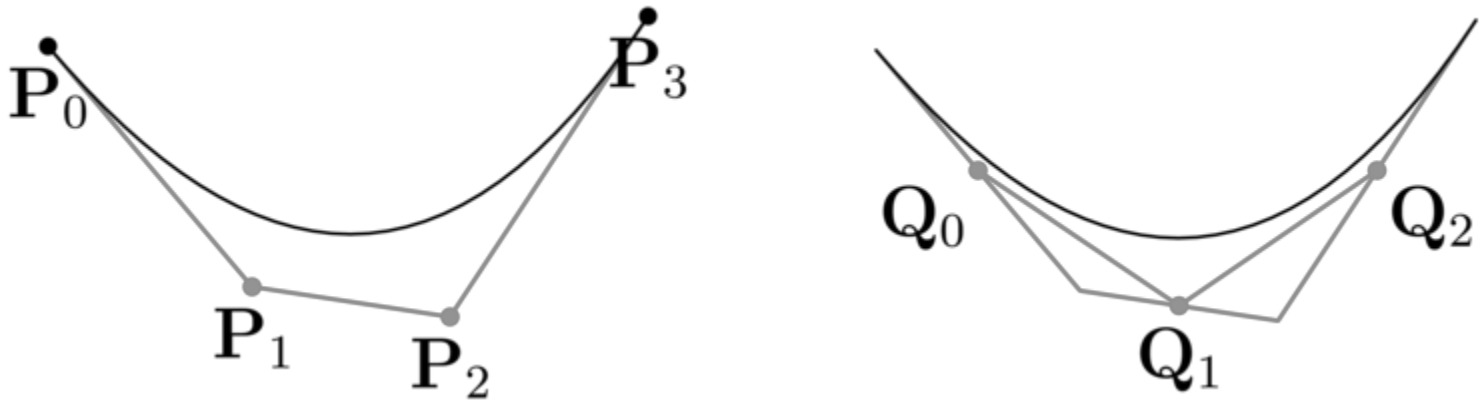
uniform tesselation

- split in K segments uniformly at $t_k = 1/k$



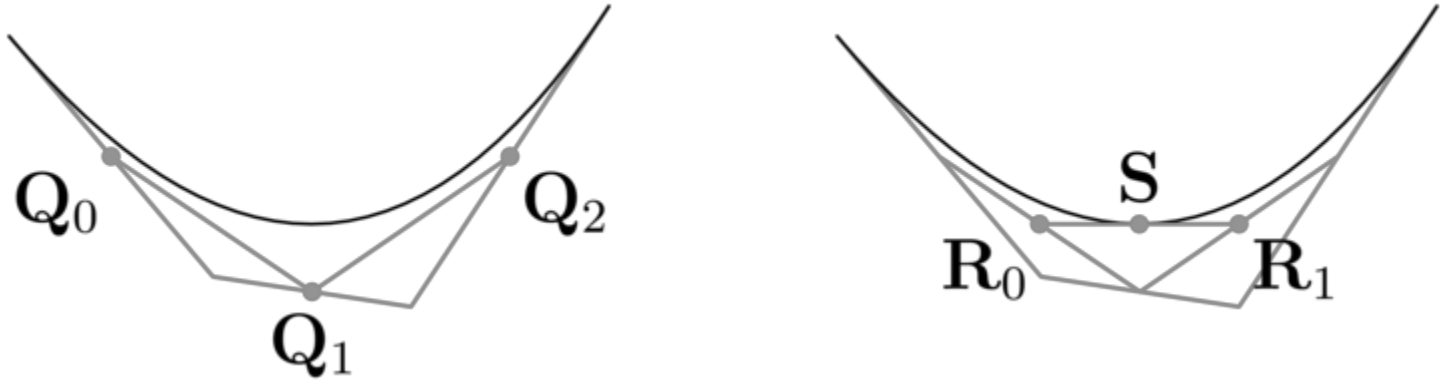
adaptive tesselation

- De Casteljau algorithm: recurdively split to small splines
  - if flat enough: draw control segments, otherwise
  - split each control segment: $\mathbf{Q}_i = (\mathbf{P}_i + \mathbf{P}_{i+1})/2$



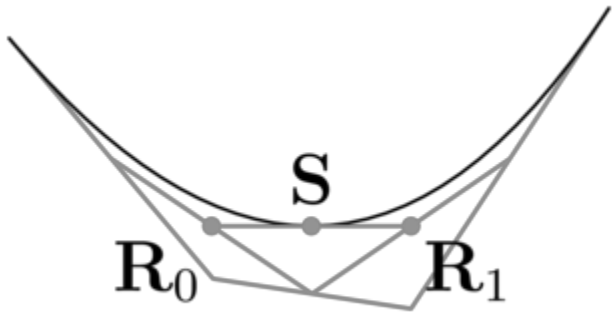
adaptive tesselation

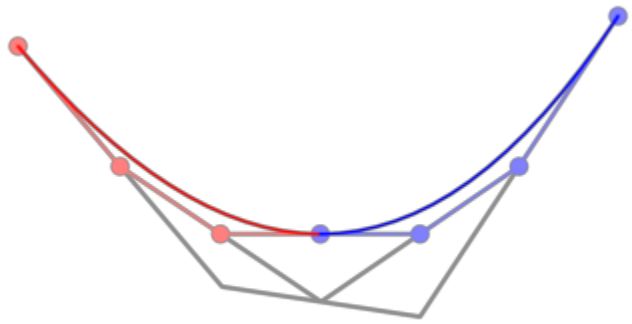
- De Casteljau algorithm: recurdively split to small splines
  - split new segments: $\mathbf{R}_i = (\mathbf{Q}_i + \mathbf{Q}_{i+1})/2$
  - split again: $\mathbf{S} = (\mathbf{R}_i + \mathbf{R}_{i+1})/2$



adaptive tesselation

- De Casteljau algorithm: recurdively split to small splines
  - two Bezier splines: $\{\mathbf{P}_0, \mathbf{Q}_0, \mathbf{R}_0, \mathbf{S}\}$, $\{\mathbf{S}, \mathbf{R}_1, \mathbf{Q}_2, \mathbf{P}_3\}$
  - recurse algorithm as above





other splines

other cubic splines types

- Bezier splines
  - most used today (2D APIs, PDF, fonts)
- Hermite splines: approximating
  - control: end points and tangents
- Catmull-Rom splines: interpolating
  - used very little
- B-splines: C^2 continuity at joints
  - impose 3 continuity constraints
- cubic splines can be converted into one another by changing control points

other spline degree

- linear and cubic most used
- can define splines for other polynomial degree
- e.g. Beziers use Bernestein polynomials of degree n

other spline functions

- uniform splines: split segments at integers
- non-uniform splines: split segments at arbitrary points
- non-uniform rational B-splines (NURBS)
  - ratios of B-splines
  - invariance under perspective
  - can represent conic sections exactly
  - often used in 3D