Rendering Equation

Physically-Based Rendering

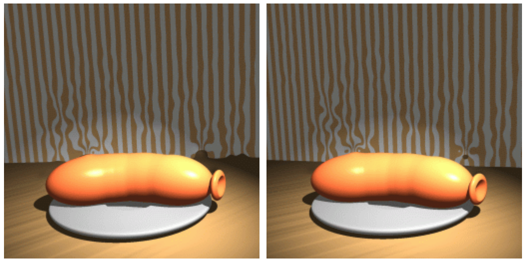
synthesis algorithms that compute images by simulation the physical behavior of light

Physically-Based Rendering

- advantages
  - predictive simulation
    - can be used for architecture, engineering, …
  - photorealistic
    - if simulation if correct, images will look real
- disadvantages
  - (really) slow
    - simulation of physics is computationally very expensive
  - need accurate geometry, materials and lights
    - otherwise just a correct solution to the wrong problem

Models of Light

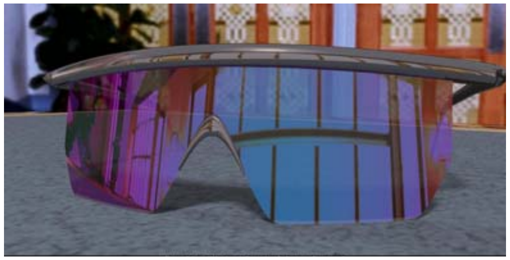
- geometric optics
  - light particles travel in straight lines
  - light particles do not interact with each other
  - describes: emission, reflection/refraction, absorption



[Stam et al., 1996]

Models of Light

- wave optics
  - light particles interact with each other
  - describes: diffraction, interference, polarization



[Gondek et al., 1997]

Models of Light

- quantum optics
  - light particles are like any other quantum particles
  - captures: fluorescence, phosphorescence

[Glassner et al., 1997]

Rendering Equation

- describe physical behavior of light in vacuum filled with objects
  - based on geometric optics principles
  - can be extended to describe participating media
  - can be extended to describe wavelenght dep.

Power and Irradiance

- power: energy per unit time
  - measured in Watts = Joules/sec

$$\Phi = \frac{dQ}{dt}$$

- irradiance: power per unit area
  - measured in Watts/meter^2

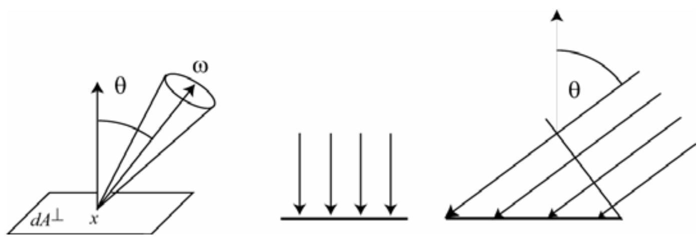
$$E = \frac{d\Phi}{dA}$$

Radiance

- power per unit projected area and solid angle
  - depends on position and direction (5D)

$$L(\mathbf{x} \to \Theta) = \frac{d^2\Phi}{dA^{\perp} d\omega_{\Theta}} = \frac{d^2\Phi}{dA \cos\theta_{\Theta} d\omega_{\Theta}} = \frac{dE}{\cos\theta_{\Theta} d\omega_{\Theta}}$$

$$\cos\theta_{\Theta} = \mathbf{N}_{\mathbf{x}} \cdot \Theta$$



[Dutré, Bekaert, Bala]

Radiance

most sensors readings (and your eyes) are proportional to radiance

Radiance Notation

- notation follows [Dutré, Bekaert, Bala]
- radiance leaving from point **x** in direction Θ

$$L(\mathbf{x} \to \Theta)$$

- radiance coming to point **x** from direction Ψ

$$L(\mathbf{x} \leftarrow \Psi)$$

- solid angle for a direction Ψ

$$d\omega_{\Psi}$$

- in general $L(\mathbf{x} \to \Theta) \neq L(\mathbf{x} \leftarrow \Theta)$

Radiance

- radiance is a function of wavelenght

$$L(\mathbf{x} \to \Theta) = \int_{\lambda \in spectrum} L(\mathbf{x} \to \Theta, \lambda)$$

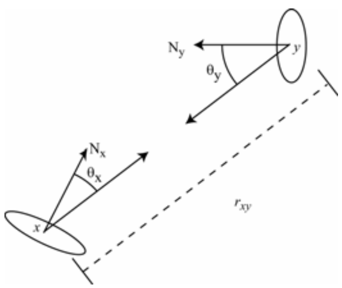
- in practice, write equations for RGB
  - we will use simplified notation without carry around the wavelength explicitly

Radiance

- formulation between two points

$$L(\mathbf{x} \to \mathbf{y}) = \frac{d^2\Phi}{dA_x \cos\theta_x d\omega_{\mathbf{x}\to\mathbf{y}}} = \frac{d^2\Phi}{dA_x dA_y} \frac{r_{xy}^2}{\cos\theta_x \cos\theta_y}$$

$$d\omega_{\mathbf{x}\to\mathbf{y}} = \frac{dA_y \cos\theta_y}{r_{xy}^2}$$



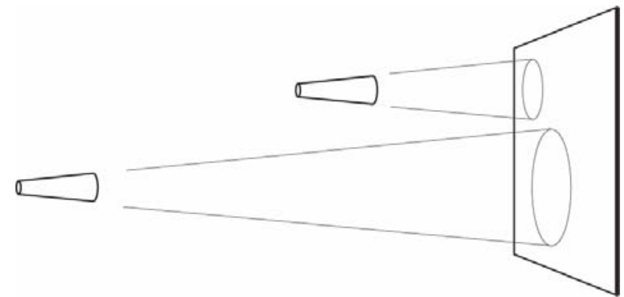
[Dutré, Bekaert, Bala]

Radiance Properties

- invariance on straight paths in vacuum
  - from energy conservation

$$L(\mathbf{x} \to \mathbf{y}) = L(\mathbf{y} \leftarrow \mathbf{x})$$

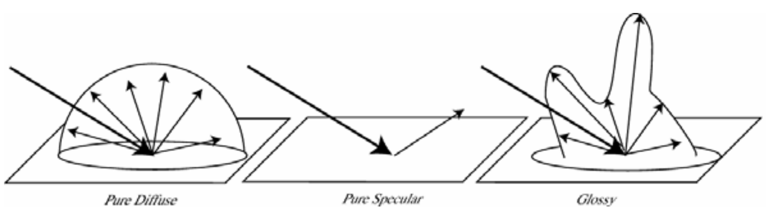
- corollary: radiance does not change with distance



[Shirley]

Material Properties

- materials differ in the way they scatter energy
  - need physical description of light scattering

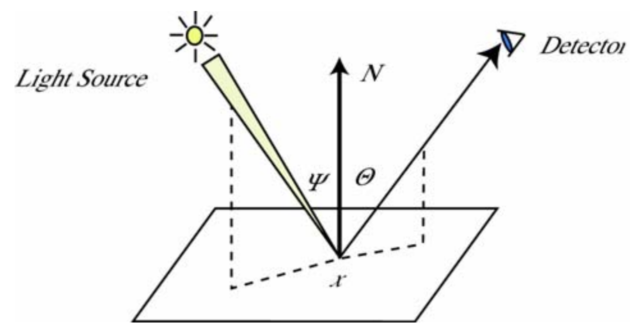


[Dutré, Bekaert, Bala]

BRDF

- bidirectional surface distribution function

$$\rho(\mathbf{x}, \Psi \to \Theta) = \frac{dL(\mathbf{x} \to \Theta)}{dE(\mathbf{x} \leftarrow \Psi)} = \frac{dL(\mathbf{x} \to \Theta)}{L(\mathbf{x} \leftarrow \Psi) d\omega_\Psi \cos\theta_\Psi}$$



[Dutré, Bekaert, Bala]

BRDF Properties

- reciprocity

$$\rho(\mathbf{x}, \Psi \to \Theta) = \rho(\mathbf{x}, \Theta \to \Psi)$$

- energy conservation

$$\forall \Psi : \int_{\Theta \in \Omega(\mathbf{x})} dL(\mathbf{x} \to \Theta) \cos\theta_\Theta d\omega_\Theta \le dE(\mathbf{x} \leftarrow \Psi) \Rightarrow$$

$$\forall \Psi : \int_{\Theta \in \Omega(\mathbf{x})} \rho(\mathbf{x}, \Psi \to \Theta) \cos\theta_\Theta d\omega_\Theta \le 1$$

Hemispherical Formulation

- need outgoing radiance in a given direction
  - from BRDF definition

$$\rho(\mathbf{x}, \Psi \to \Theta) = \frac{dL(\mathbf{x} \to \Theta)}{L(\mathbf{x} \leftarrow \Psi) d\omega_\Psi \cos\theta_\Psi}$$

  - determine reflected radiance L_r by integration over all incoming light

$$L_r(\mathbf{x} \to \Theta) = \int dL(\mathbf{x} \to \Theta) =$$

$$= \int_{\Psi \in \Omega_\mathbf{x}} L(\mathbf{x} \leftarrow \Psi) \rho(\mathbf{x}, \Psi \to \Theta) \cos\theta_\Psi d\omega_\Psi$$

Hemispherical Formulation

- need outgoing radiance in a given direction
  - also consider light spontaneously emitted by surface

$$L_e(\mathbf{x} \to \Theta)$$

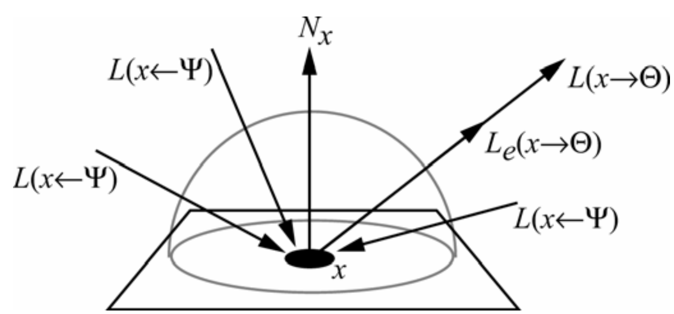
  - total radiance is the sum of emitted and reflected

$$L(\mathbf{x} \to \Theta) = L_e(\mathbf{x} \to \Theta) + L_r(\mathbf{x} \to \Theta)$$

$$L(\mathbf{x} \to \Theta) = L_e(\mathbf{x} \to \Theta) + \int_{\Psi \in \Omega_\mathbf{x}} L(\mathbf{x} \leftarrow \Psi) \rho(\mathbf{x}, \Psi \to \Theta) \cos(\mathbf{N}_\mathbf{x}, \Psi) d\omega_\Psi$$

Hemispherical Formulation

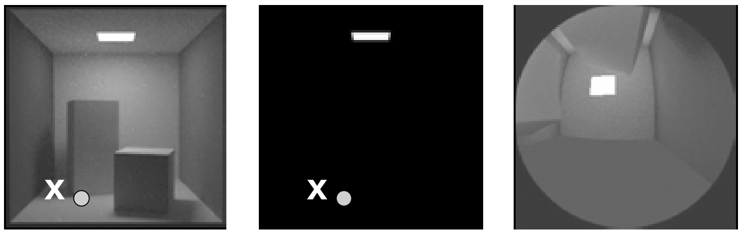
$$L(\mathbf{x} \to \Theta) = L_e(\mathbf{x} \to \Theta) + \int_{\Psi \in \Omega_\mathbf{x}} L(\mathbf{x} \leftarrow \Psi) \rho(\mathbf{x}, \Psi \to \Theta) \cos\theta_\Psi d\omega_\Psi$$



[Dutré, Bekaert, Bala]

Intuition Behind Rendering Equation

$$L(\mathbf{x} \to \Theta) = L_e(\mathbf{x} \to \Theta) + \\ + \int_{\Psi \in \Omega_\mathbf{x}} L(\mathbf{x} \leftarrow \Psi)\rho(\mathbf{x}, \Psi \to \Theta)\cos\theta_\Psi d\omega_\Psi$$



$L(\mathbf{x} \to \Theta)$ $L_e(\mathbf{x} \to \Theta)$ $L(\mathbf{x} \leftarrow \Psi)$

[Bala]

Intuition Behind Rendering Equation

$$L(\mathbf{x} \to \Theta) = L_e(\mathbf{x} \to \Theta) + \\ + \int_{\Psi \in \Omega_\mathbf{x}} L(\mathbf{x} \leftarrow \Psi)\rho(\mathbf{x}, \Psi \to \Theta)\cos\theta_\Psi d\omega_\Psi$$

integral equation

indicates radiance at equilibrium

Visible Point Formulation

- point visible from x in direction Ψ

$$\mathbf{y} = r(\mathbf{x}, \Psi)$$

- since energy is conserved in vacuum

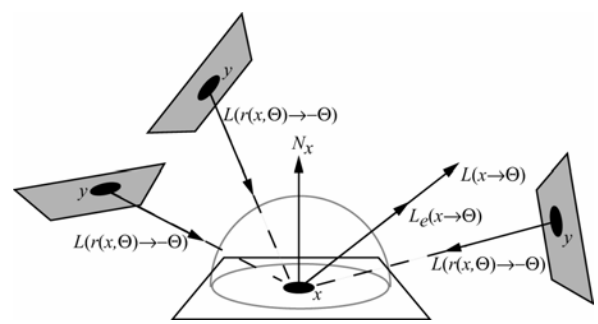
$$L(\mathbf{x} \leftarrow \Psi) = L(\mathbf{y} \to -\Psi)$$

- by substituting previous values in rendering eq.

$$L(\mathbf{x} \to \Theta) = L_e(\mathbf{x} \to \Theta) + \\ + \int_{\Psi \in \Omega_\mathbf{x}} L(r(\mathbf{x}, \Psi) \to -\Psi)\rho(\mathbf{x}, \Psi \to \Theta)\cos\theta_\Psi d\omega_\Psi$$

Visible Point Formulation

$$L(\mathbf{x} \to \Theta) = L_e(\mathbf{x} \to \Theta) + \\ + \int_{\Psi \in \Omega_\mathbf{x}} L(r(\mathbf{x}, \Psi) \to -\Psi)\rho(\mathbf{x}, \Psi \to \Theta)\cos\theta_\Psi d\omega_\Psi$$

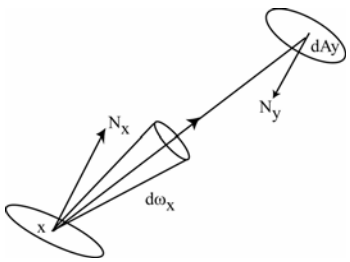


[Dutré, Bekaert, Bala]

Area Formulation

- compute solid angle visible from x to y

$$d\omega_\Psi = \frac{dA_{\mathbf{y}} \cos\theta_{\mathbf{y}}}{r_{\mathbf{xy}}^2}$$



[Dutré, Bekaert, Bala]

Area Formulation

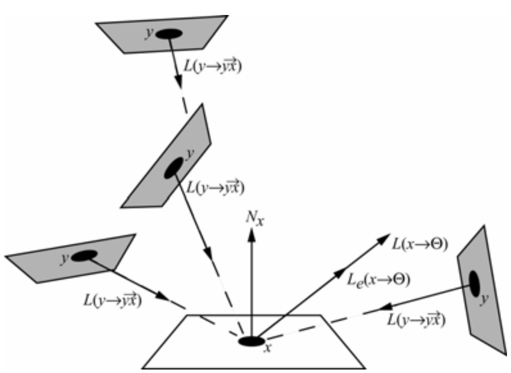
- by changing domain from hemisphere to scene
  - and introducing explicit visibility evaluation V

$$L(\mathbf{x} \to \Theta) = L_e(\mathbf{x} \to \Theta) + \\ + \int_{\mathbf{y}\in S} L(\mathbf{y} \to \overrightarrow{\mathbf{yx}}) \rho(\mathbf{x}, \Psi \to \Theta) \frac{\cos\theta_\Psi \cos\theta_\Theta}{r_{\mathbf{xy}}^2} V(\mathbf{x},\mathbf{y}) dA_{\mathbf{y}}$$

$$G(\mathbf{x},\mathbf{y}) = \frac{\cos\theta_\Psi \cos\theta_\Theta}{r_{\mathbf{xy}}^2} = \frac{(\overrightarrow{\mathbf{xy}} \cdot \mathbf{N}_{\mathbf{x}})(\overrightarrow{\mathbf{yx}} \cdot \mathbf{N}_{\mathbf{y}})}{|\mathbf{x} - \mathbf{y}|^2}$$

Area Formulation

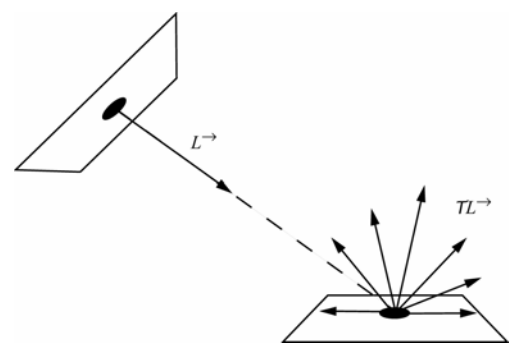
$$L(\mathbf{x} \to \Theta) = L_e(\mathbf{x} \to \Theta) + \\ + \int_{\mathbf{y}\in S} L(\mathbf{y} \to \overrightarrow{\mathbf{yx}}) \rho(\mathbf{x}, \Psi \to \Theta) G(\mathbf{x},\mathbf{y}) V(\mathbf{x},\mathbf{y}) dA_{\mathbf{y}}$$



[Dutré, Bekaert, Bala]

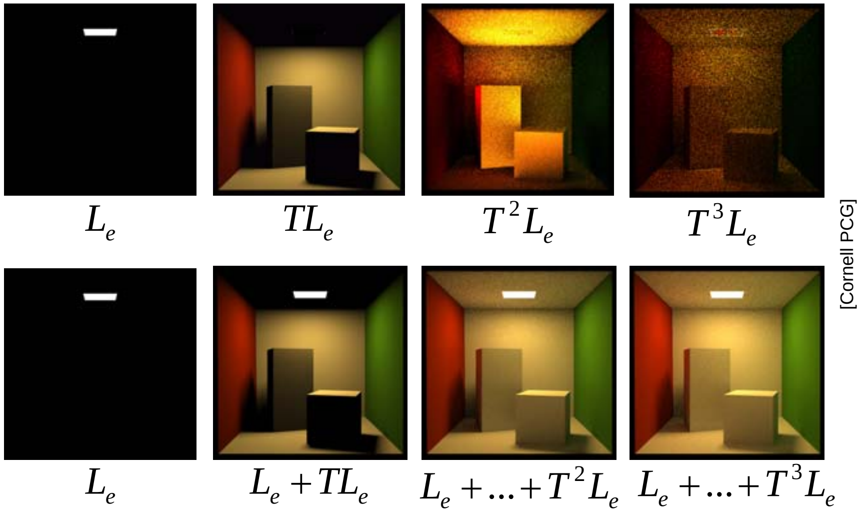
Transport Formulation

$$L = L_e + TL \Rightarrow L_e + TL_e + TTL_e + \ldots = \sum_{i=0}^{\infty} T^i L_e$$



[Dutré, Bekaert, Bala]

Transport Formulation



Direct and Indirect Illum. Formulation

- direct illumination: radiance reaching a surface directly from the light
  - often efficient to sample using area formulation
- indirect illumination: radiance reaching a surface after bouncing at least once on another surface
  - often efficient to sample using hemisphere formulation

Direct and Indirect Illum. Formulation

$$L(\mathbf{x} \to \Theta) = L_e(\mathbf{x} \to \Theta) + L_r(\mathbf{x} \to \Theta)$$

$$L_r(\mathbf{x} \to \Theta) = \int L(\mathbf{x} \leftarrow \Psi)\rho(\ldots)\cos\theta_\Psi d\omega_\Psi =$$

$$= \int \left(L_e(\mathbf{x} \leftarrow \Psi) + L_r(\mathbf{x} \leftarrow \Psi)\right)\rho(\ldots)\cos\theta_\Psi d\omega_\Psi =$$

$$= L_d(\mathbf{x} \to \Theta) + L_i(\mathbf{x} \to \Theta)$$

Direct Illumination Formulation

$$L_d(\mathbf{x} \to \Theta) = \int_{\Psi \in \Omega_\mathbf{x}} L_e(\mathbf{x} \leftarrow \Psi)\rho(\mathbf{x}, \Psi \to \Theta)\cos\theta_\Psi d\omega_\Psi$$

rewrite in area formulation

$$L_d(\mathbf{x} \to \Theta) = \int_{\mathbf{y} \in lights\ surface} L_e(\mathbf{x} \leftarrow \overrightarrow{\mathbf{yx}})\rho(\mathbf{x}, \overrightarrow{\mathbf{xy}} \to \Theta)G(\mathbf{x},\mathbf{y})V(\mathbf{x},\mathbf{y})dA_\mathbf{y}$$

$$L_d(\mathbf{x} \to \Theta) = \sum_l \int_{\mathbf{y} \in light\ l} L_e(\mathbf{x} \leftarrow \overrightarrow{\mathbf{yx}})\rho(\mathbf{x}, \overrightarrow{\mathbf{xy}} \to \Theta)G(\mathbf{x},\mathbf{y})V(\mathbf{x},\mathbf{y})dA_\mathbf{y}$$

Indirect Illumination Formulation

$$L_d(\mathbf{x} \to \Theta) = \int_{\Psi \in \Omega_\mathbf{x}} L_r(\mathbf{x} \leftarrow \Psi)\rho(\mathbf{x}, \Psi \to \Theta)\cos\theta_\Psi \, d\omega_\Psi$$

since $L_r(\mathbf{x} \leftarrow \Psi) = L_r(r(\mathbf{x}, \Psi) \to -\Psi)$

$$L_d(\mathbf{x} \to \Theta) = \int_{\Psi \in \Omega_\mathbf{x}} L_r(r(\mathbf{x}, \Psi) \to -\Psi)\rho(\mathbf{x}, \Psi \to \Theta)\cos\theta_\Psi \, d\omega_\Psi$$

Hemispherical Integration

- 2D square

$$I = \int_{\mathbf{x} \in S} f(\mathbf{x}) dA_\mathbf{x} = \int_{x_0}^{x_1} \int_{y_0}^{y_1} f(x, y) dx dy$$

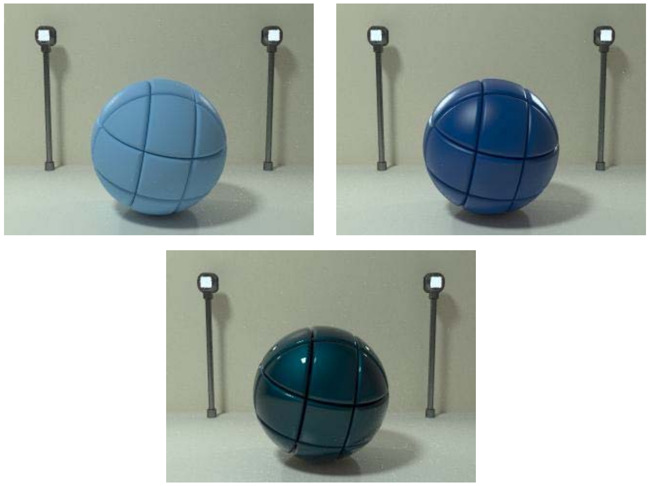
- 2D hemisphere

$$I = \int_{\Theta \in \Omega} f(\Theta) d\omega_\Theta = \int_0^{2\pi} \int_0^{\pi} f(\varphi, \theta) \sin\theta d\varphi d\theta$$

Materials

Physically-Based Materials

- capture realistic appearance is necessary

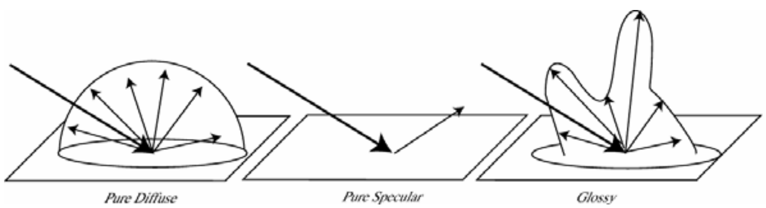


[Cornell PCG]

Diffuse BRDF

- light is reflected equally in all directions

$$\rho(\mathbf{x}, \Psi \to \Theta) = \frac{\rho_d}{\pi}$$



[Dutré, Bekaert, Bala]

Diffuse BRDF

- Lambertian shading model motivation

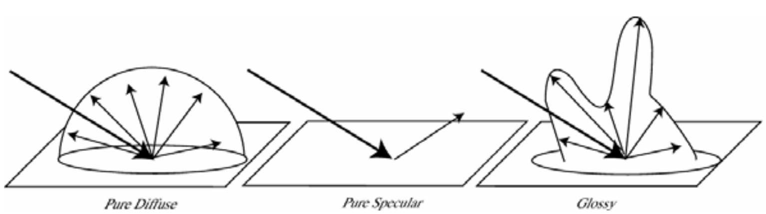
$$dL(\mathbf{x} \to \Theta) = \rho(\mathbf{x}, \Psi \to \Theta) dE(\mathbf{x} \leftarrow \Psi) =$$

$$= \frac{\rho_d}{\pi} \cos\theta_\Psi L(\mathbf{x} \leftarrow \Psi) d\omega_\Psi = C_l k_d \cos\theta$$

Specular BRDF

- light is reflected only in one direction

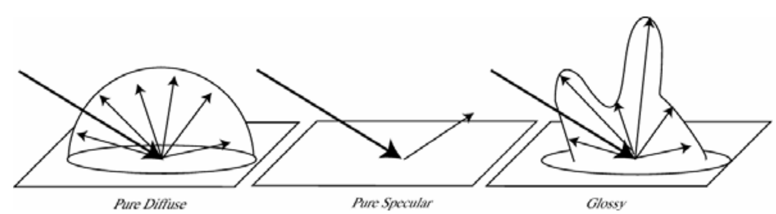
$$\rho(\mathbf{x}, \Psi \to \Theta) \propto \delta(\Psi, \Theta)$$



[Dutré, Bekaert, Bala]

Glossy BRDFs

- light is reflected in many directions unequally
  - many models exist



[Dutré, Bekaert, Bala]

Glossy BRDFs – Phong and Blinn Models

- Phong model

$$\rho(\mathbf{x},\Psi \to \Theta) = k_d + k_s \cos^n \theta_r = k_d + k_s (\mathbf{R}\cdot\Theta)^n$$

- Blinn-Phong model

$$\rho(\mathbf{x},\Psi \to \Theta) = k_d + k_s (\mathbf{N}\cdot\mathbf{H})^n$$

- issues:
  - non reciprocal
  - non energy conserving

Glossy BRDFs – Modified Blinn-Phong Model

- modified Blinn-Phong model

$$\rho(\mathbf{x},\Psi \to \Theta) = \frac{\rho_d}{\pi} + \frac{n+2}{2\pi}\rho_s (\mathbf{H}\cdot\Theta)^n$$

- energy conservation

$$\rho_d + \rho_s \le 1$$

Glossy BRDFs – Modified Phong Model

- is modified Phong physically accurate?

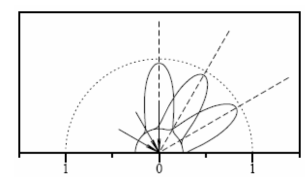
photograph | Phong | accurate BRDF

[LaFortune et al., 1997]

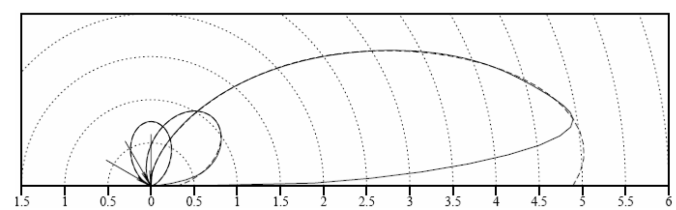
Glossy BRDFs – Modified Phong Model

- is modified Phong physically accurate?

Phong



accurate BRDF



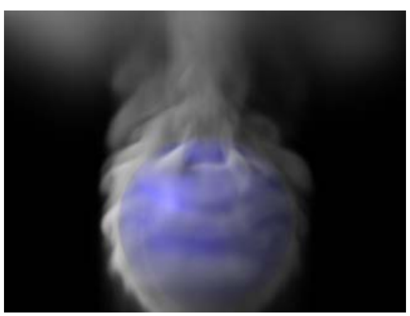
[LaFortune et al., 1997]

Glossy BRDFs - Better Models

- analytic models
  - physically motivated
  - hard to capture every material
- data-driven
  - measure light reflectance
  - encode in lookup table or fit
  - resample when rendering

Extending the Rendering Equation

Participating Media



[Fedkiw et al.]

Subsurface Scattering

BRDF BSSRDF

[Jensen et al.]

Subsurface Scattering

(a) 3D mesh (close-up of nostril) (b) Color data (c) Diffuse rendering

(d) Oily layer (e) Subsurface scattering (f) Final result

[Jensen]

Subsurface Scattering

[Jensen et al.]