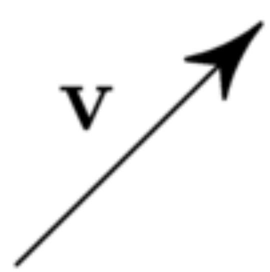
math review

vector math

vector math

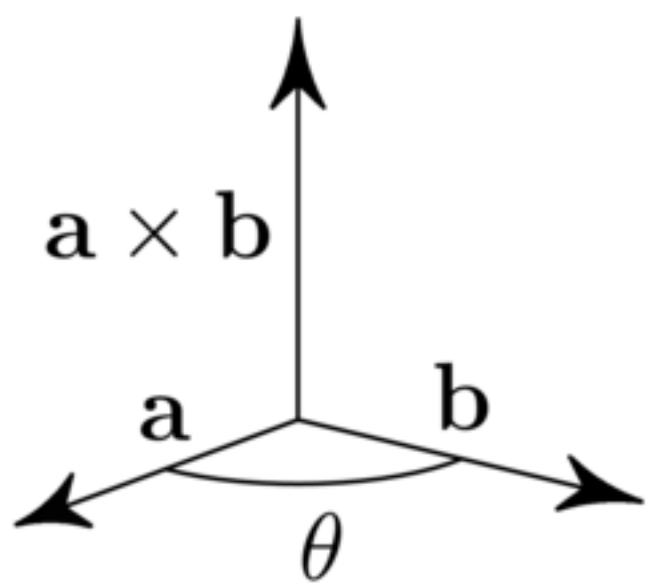
- point: location in 3D space
  - $\mathbf{P} = (P_x, P_y, P_z)$

- vector: direction and magnitude
  - $\mathbf{v} = (v_x, v_y, v_z)$



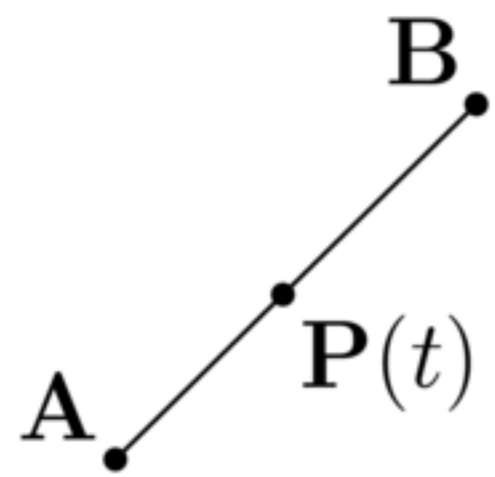
vector math

- dot product
  - $\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}||\mathbf{b}| \cos\theta$
- cross product
  - $|\mathbf{a} \times \mathbf{b}| = |\mathbf{a}||\mathbf{b}| \sin\theta$
  - $\mathbf{a} \times \mathbf{b}$ is orthogonal to $\mathbf{a}$ and $\mathbf{b}$



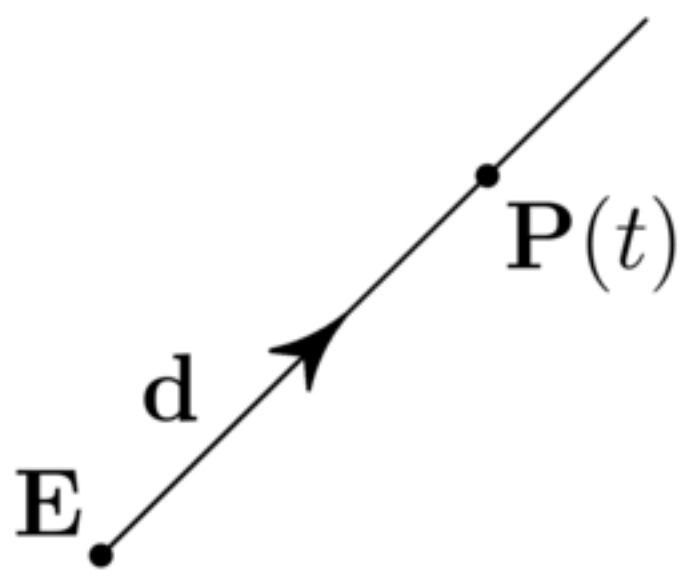
vector math

- segment: set of points (line) between two points
  - $\mathbf{P}(t) = \mathbf{A} + t(\mathbf{B} - \mathbf{A})$ with $t \in [0, 1]$



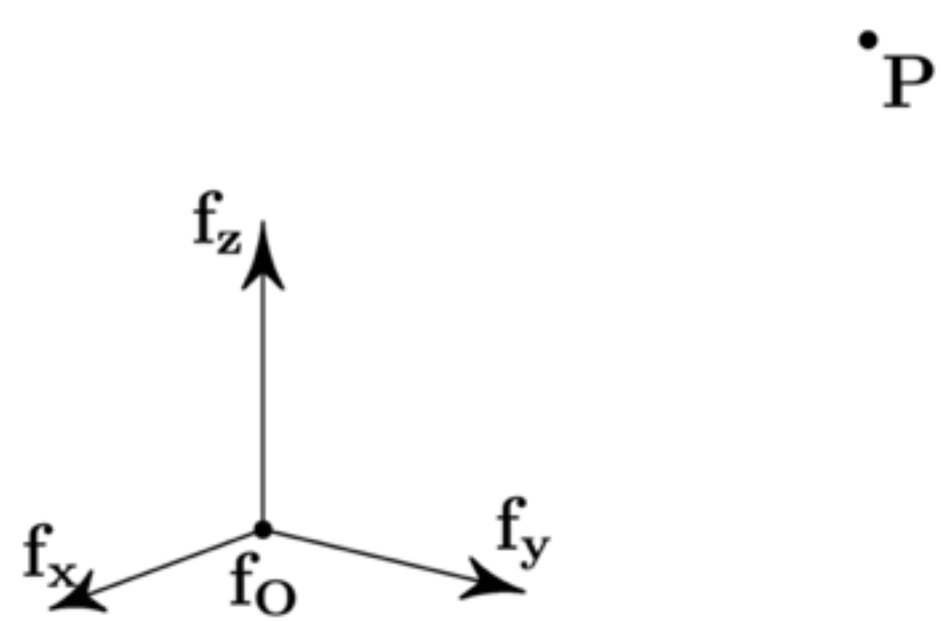
vector math

- ray: infinite line from point in a given direction
    - $\mathbf{P}(t) = \mathbf{E} + t\mathbf{d}$ with $t \in [0, \infty]$



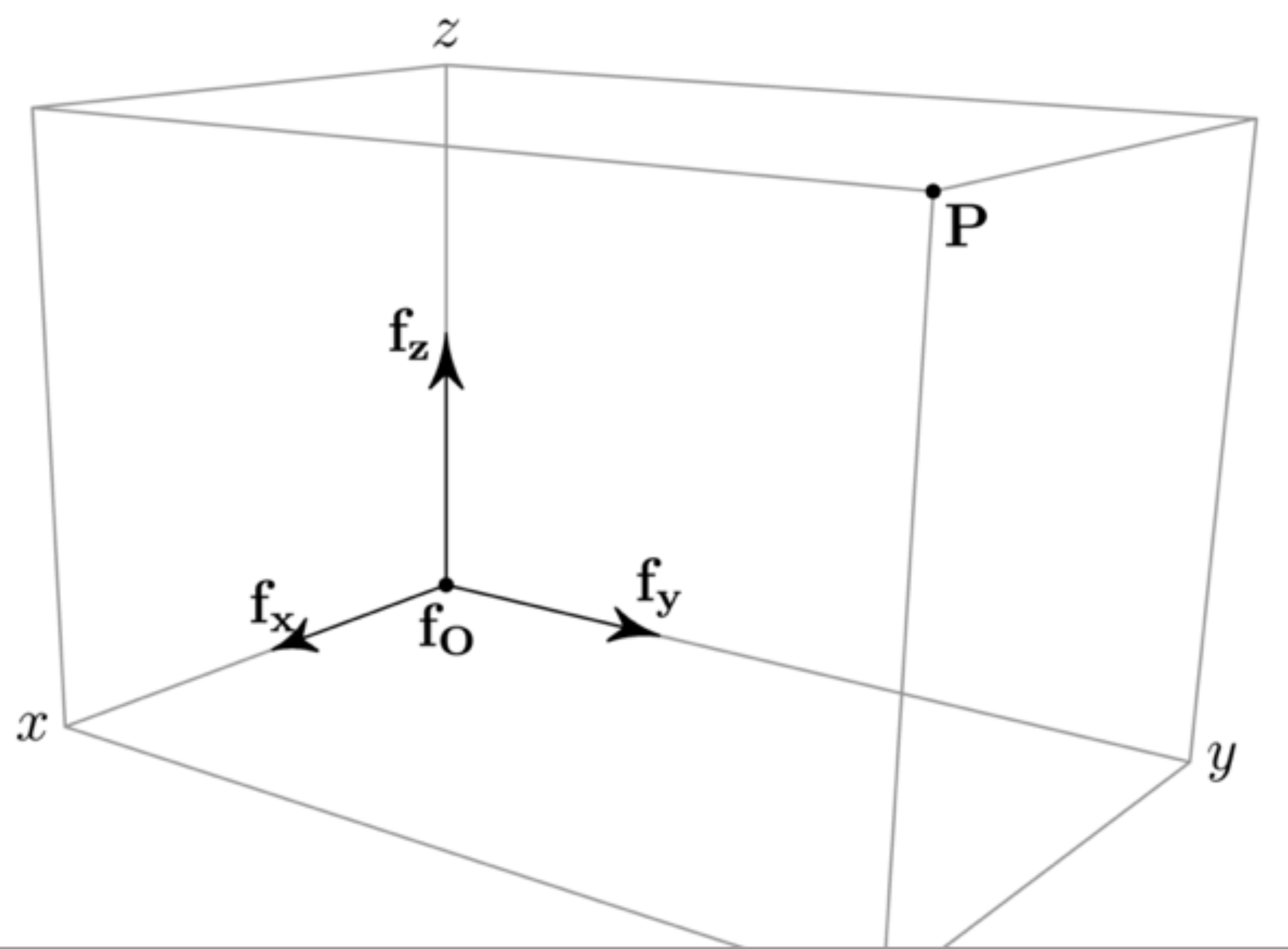
vector math

- coordinate system aka frame
  - frame $\mathbf{f} = \{\mathbf{f_O}, \mathbf{f_x}, \mathbf{f_y}, \mathbf{f_z}\}$: position and orthonormal axes
  - default (or *world*) frame: origin and three major axes



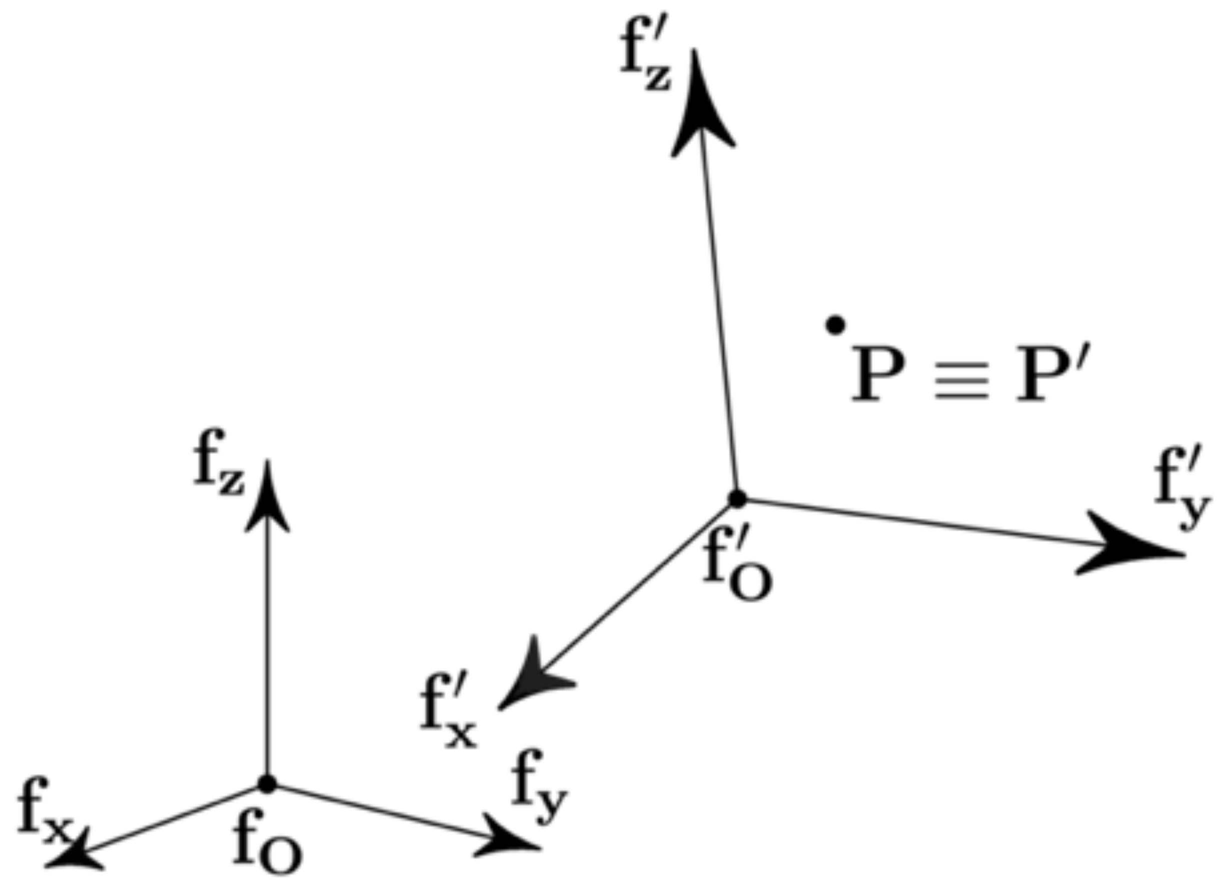
vector math

- point coords are defined wrt a frame
  - $\mathbf{P} = (P_x, P_y, P_z)$ wrt $\{\mathbf{f_O}, \mathbf{f_x}, \mathbf{f_y}, \mathbf{f_z}\}$ (*world* if not specified)
  - $\mathbf{P} = \left((\mathbf{P} - \mathbf{f_O}) \cdot \mathbf{f_x}, (\mathbf{P} - \mathbf{f_O}) \cdot \mathbf{f_y}, (\mathbf{P} - \mathbf{f_O}) \cdot \mathbf{f_z}\right)$



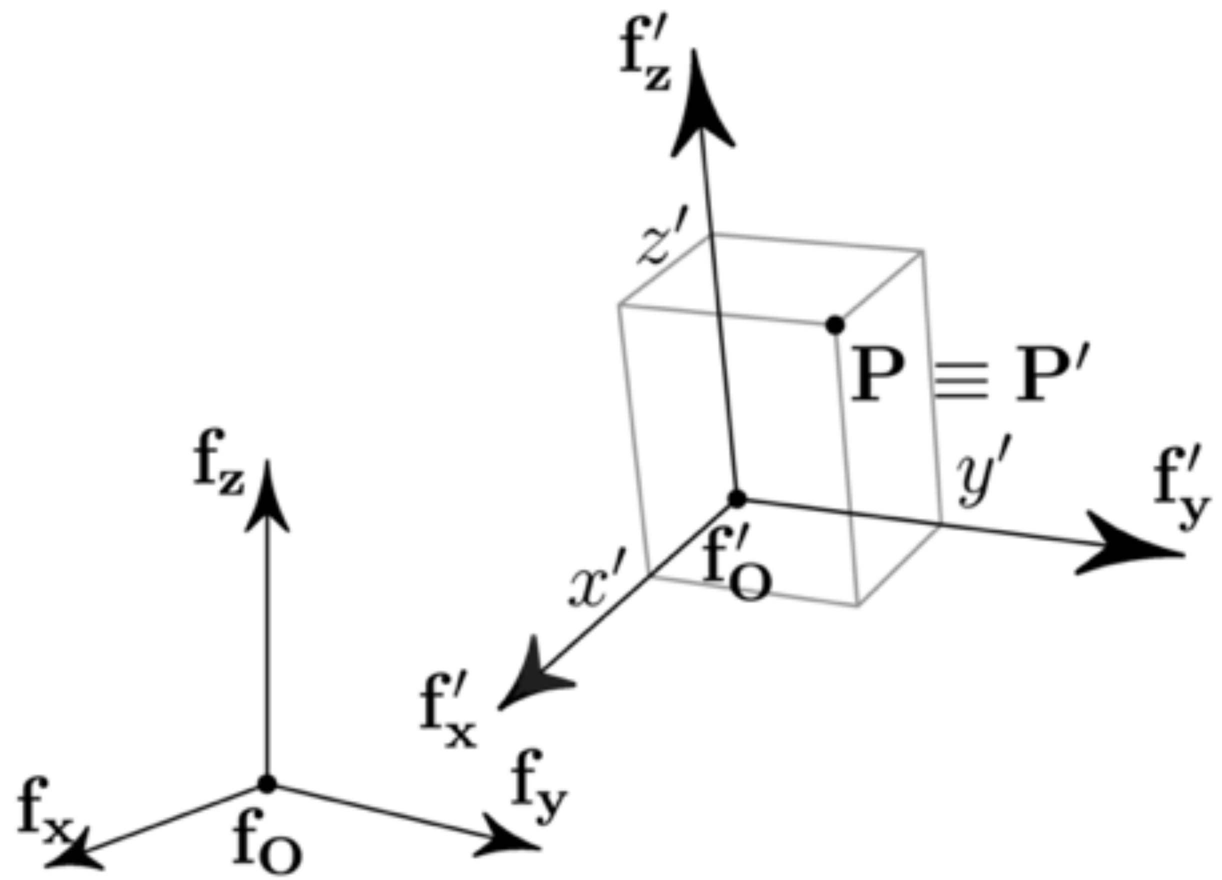
vector math

- change of coordinate system $\mathbf{f} \to \mathbf{f}'$
  - $\mathbf{P}' = (P'_x, P'_y, P'_z)$ is $\mathbf{P}$ w.r.t $\{\mathbf{f}'_\mathbf{O}, \mathbf{f}'_\mathbf{x}, \mathbf{f}'_\mathbf{y}, \mathbf{f}'_\mathbf{z}\}$
  - $\mathbf{P}' = \left((\mathbf{P} - \mathbf{f}'_\mathbf{O}) \cdot \mathbf{f}'_\mathbf{x}, (\mathbf{P} - \mathbf{f}'_\mathbf{O}) \cdot \mathbf{f}'_\mathbf{y}, (\mathbf{P} - \mathbf{f}'_\mathbf{O}) \cdot \mathbf{f}'_\mathbf{z}\right)$



vector math

- change of coordinate system $\mathbf{f}' \to \mathbf{f}$
  - $\mathbf{P}' = (P'_x, P'_y, P'_z)$ is $\mathbf{P}$ w.r.t $\{\mathbf{f}'_\mathbf{O}, \mathbf{f}'_\mathbf{x}, \mathbf{f}'_\mathbf{y}, \mathbf{f}'_\mathbf{z}\}$
  - $\mathbf{P} = \mathbf{f}'_\mathbf{O} + P'_x\mathbf{f}'_\mathbf{x} + P'_y\mathbf{f}'_\mathbf{y} + P'_z\mathbf{f}'_\mathbf{z}$



vector math

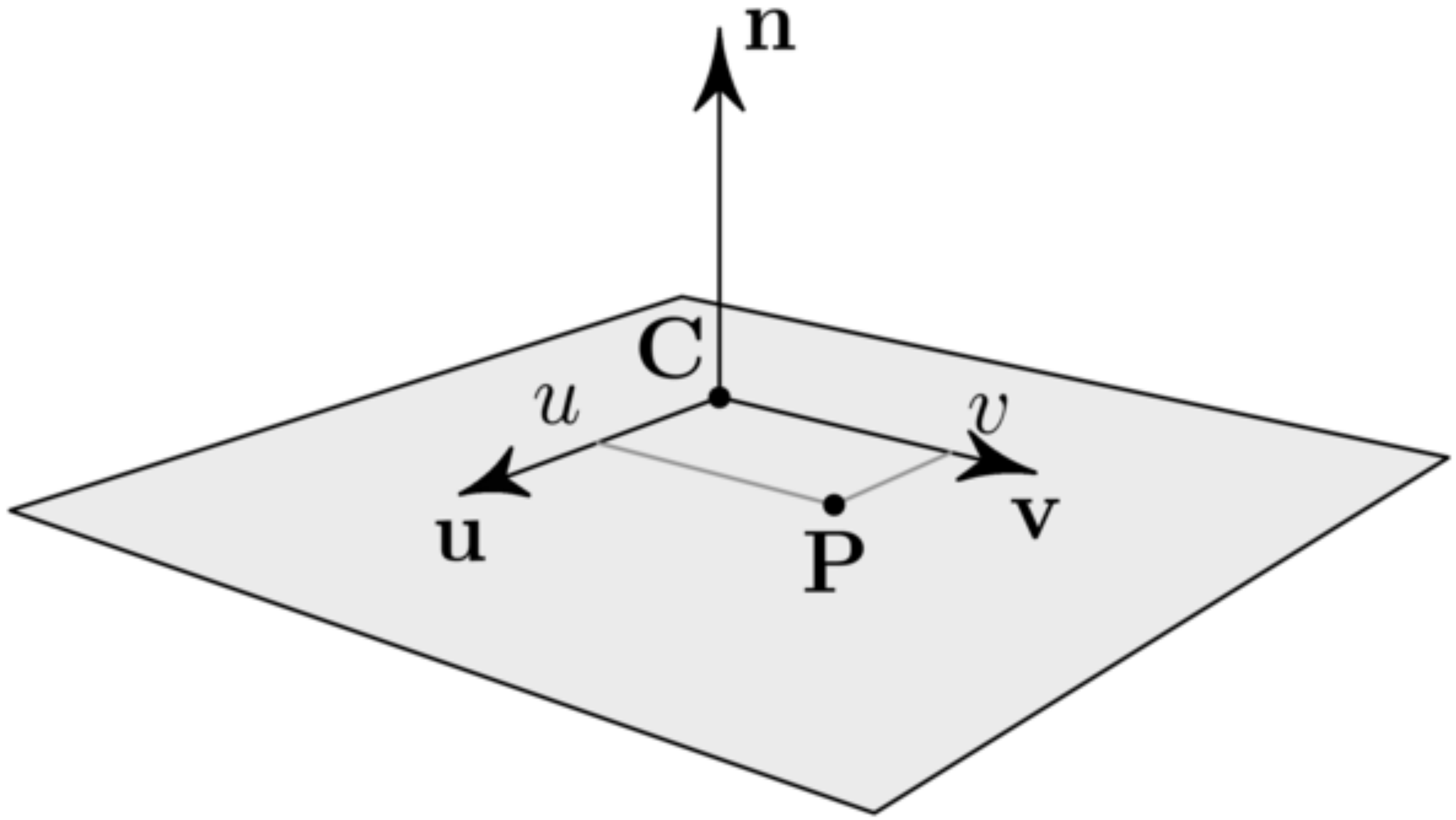
- vector coords are defined wrt a frame
  - to change coord system, ignore origin
  - $\mathbf{v} = v'_x \mathbf{f}'_\mathbf{x} + v'_y \mathbf{f}'_\mathbf{y} + v'_z \mathbf{f}'_\mathbf{z}$
  - $\mathbf{v}' = \left(\mathbf{v} \cdot \mathbf{f}'_\mathbf{x}, \mathbf{v} \cdot \mathbf{f}'_\mathbf{y}, \mathbf{v} \cdot \mathbf{f}'_\mathbf{z}\right)$

vector math

- construct a frame from two non-orthonormal vectors $\mathbf{z}'$, $\mathbf{y}'$
  - assume that $\mathbf{z}'$ is not parallel to $\mathbf{y}'$
  - $\mathbf{z} = \mathbf{z}' / |\mathbf{z}'|$
  - $\mathbf{x} = \mathbf{y}' \times \mathbf{z} / |\mathbf{y}' \times \mathbf{z}|$
  - $\mathbf{y} = \mathbf{z} \times \mathbf{x}$
- construct a frame from a vector $\mathbf{z}'$
  - pick arbitrary $\mathbf{y}'$ and continue as above

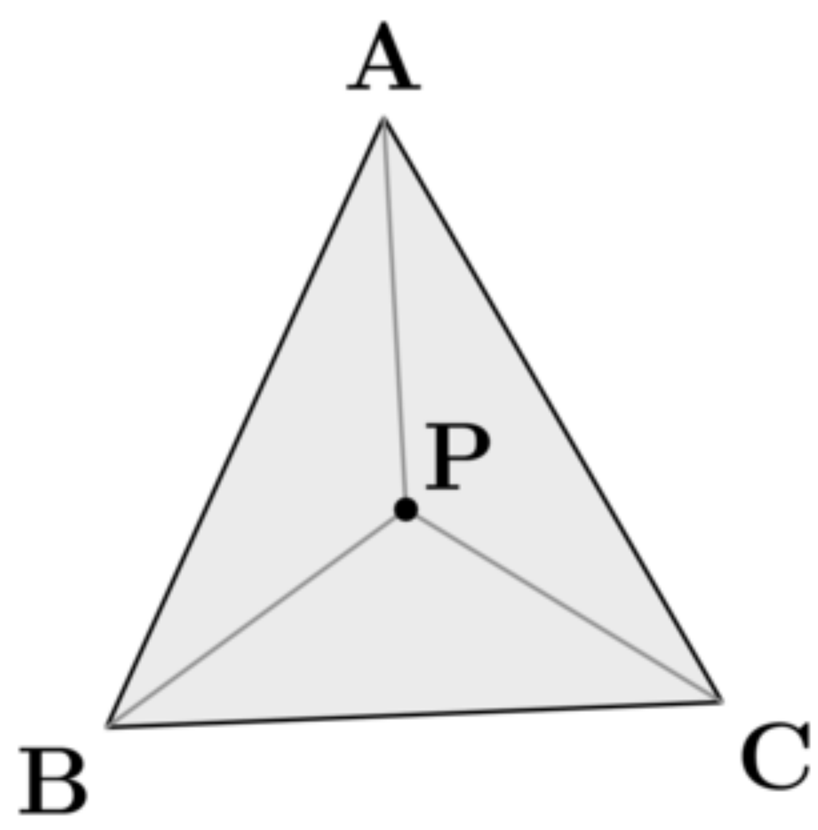
vector math

- infinite plane
  - $\mathbf{P} \in plane \iff (\mathbf{P} - \mathbf{C}) \cdot \mathbf{n} = 0 \iff \mathbf{P} \cdot \mathbf{n} = d$
  - $\mathbf{P}(u, v) = \mathbf{C} + u \cdot \mathbf{u} + v \cdot \mathbf{v}$ with $(u, v) \in (-\infty, \infty)^2$
  - normal: $\mathbf{n} = \mathbf{u} \times \mathbf{v}$



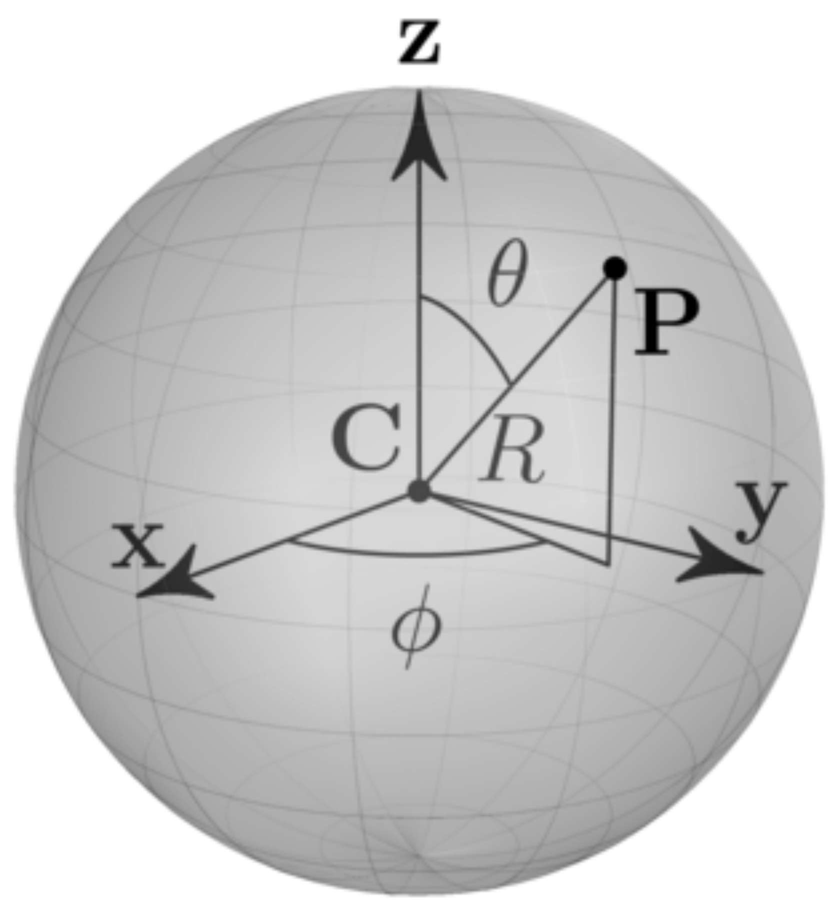
vector math

- triangle baricentric coordinates
  - $\mathbf{P}(\alpha, \beta, \gamma) = \alpha\mathbf{A} + \beta\mathbf{B} + \gamma\mathbf{C}$ with $\alpha + \beta + \gamma = 1$
  - $\mathbf{P}(\alpha, \beta) = \alpha(\mathbf{A} - \mathbf{C}) + \beta(\mathbf{B} - \mathbf{C}) + \mathbf{C}$
  - $\alpha = area\left(\mathbf{BCP}\right)/area\left(\mathbf{ABC}\right)$, ...



vector math

- sphere
    - $\mathbf{P} \in sphere \iff |\mathbf{P} - \mathbf{C}| = R$
    - $\mathbf{P}(u, v) = \mathbf{C} + R \cdot \left(\cos\phi \sin\theta, \sin\phi \sin\theta, \cos\theta\right)$



linear algebra review

matrices

- matrix and vector notation
  - use column for vectors

$$M = \begin{bmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{bmatrix} = \begin{bmatrix} m_{ij} \end{bmatrix}$$

$$\mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} v_1 & v_2 \end{bmatrix}^T$$

matrix operations

- addition

$$T = M + N$$

$$\left[t_{ij} \right] = \left[m_{ij} + n_{ij} \right]$$

- scalar multiplication

$$T = aM$$

$$\left[t_{ij} \right] = \left[a m_{ij} \right]$$

matrix operations

- matrix-matrix multiplication
  - row-column multiplication
  - not commutative
  - associative

$$T = MN = \left[t_{ij} \right] = \left[\sum_k m_{ik} n_{kj} \right]$$

$$\begin{bmatrix} t_{11} & t_{12} \\ t_{21} & t_{22} \end{bmatrix} = \begin{bmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{bmatrix} \begin{bmatrix} n_{11} & n_{12} \\ n_{21} & n_{22} \end{bmatrix}$$

matrix operations

- matrix-vector multiplication
  - row-column multiplication

$$\mathbf{u} = M\mathbf{v}$$

$$\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$$

matrix operations

- transpose: flip along diagonal

$$T = M^T$$

$$\left[t_{ij} \right] = \left[t_{ji} \right]$$

- inverse (not computed explicitly in this course)

$$T = M^{-1}$$

$$MT = MM^{-1} = M^{-1}M = I$$

special matrices

- identity: invariant for multiplication

$$I = \left[i_{ij} \right] = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}$$

$$I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\forall M : M = MI = IM$$

special matrices

- zero: invariant for addition

$$O = \left[i_{ij} \right] = 0$$

$$I = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$\forall M : M = M + O = O + M$$

matrix operation properties

- linearity of multiplication and addition

$$a(A + B) = aA + aB$$

$$M(aA + bB) = aMA + bMB$$

- associativity of multiplication

$$A(BC) = (AB)C$$

matrix operation properties

- transpose and inverse of multiplication

$$(AB)^T = B^T A^T$$

$$(AB)^{-1} = B^{-1} A^{-1}$$