parametric spline curves

curves

- used in many contexts
  - fonts
  - animation paths
  - shape modeling

- different representation
  - implicit curves
  - parametric curves
    - mostly used

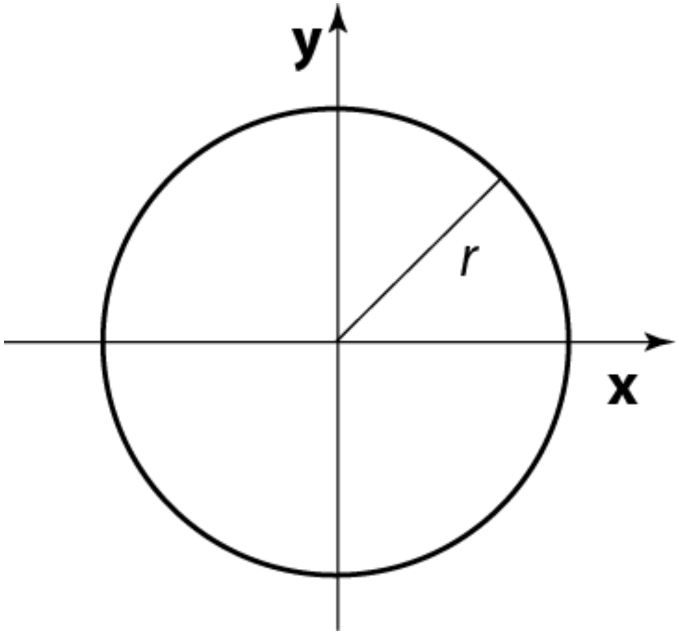
implicit representation for 2D curves

- curves can be represented implicitly as

$$f(\mathbf{p}) = f(x, y) = 0$$

- example: circle of radius *r* centered at origin

$$x^2 + y^2 - r^2 = 0$$



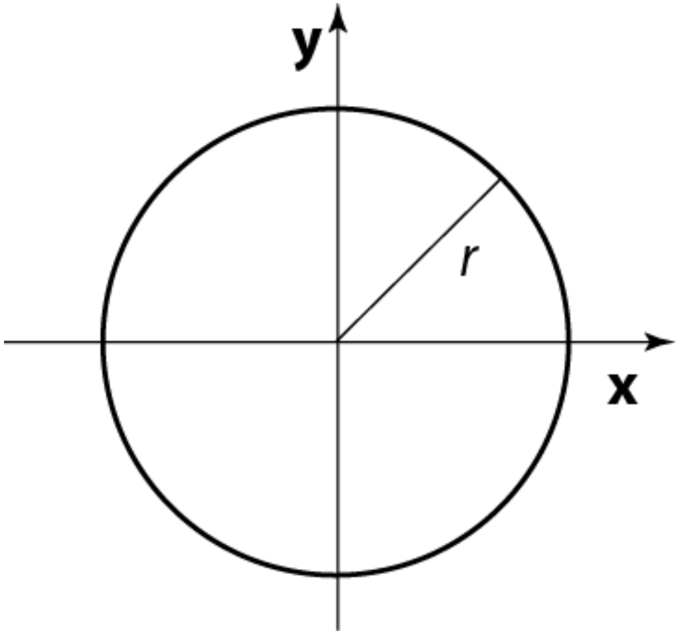
parametric representation for 2D curves

- curves can be represented parametrically as

$$\mathbf{p}(u) = \begin{cases} x = f_x(u) \\ y = f_y(u) \end{cases}$$

- example: circle of radius *r* centered at origin

$$\begin{cases} x = r\cos(u) \\ y = r\sin(u) \end{cases}$$



parametric representation of splines

- general parametric curve can be written as

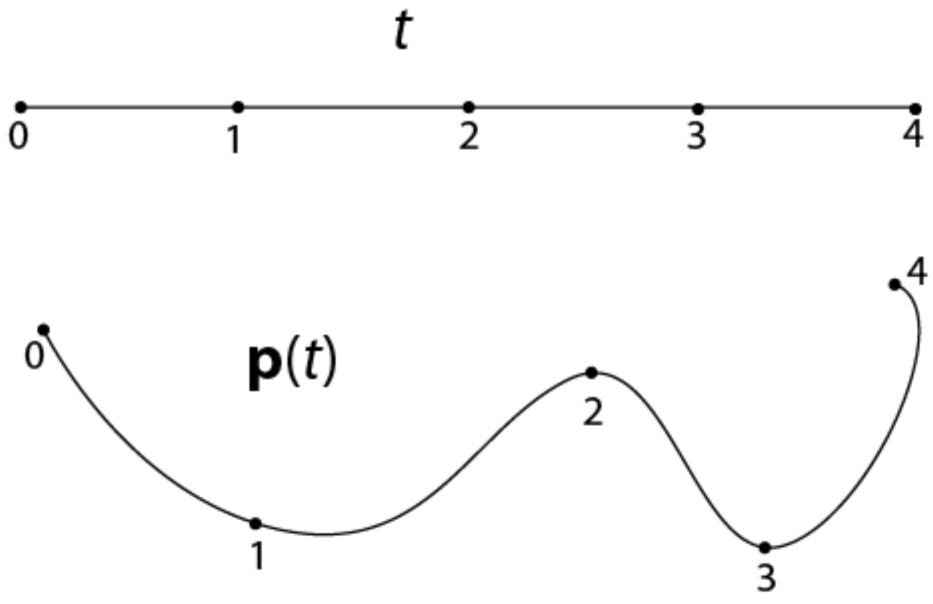
$$\mathbf{p}(t) = f(t) \qquad t \in [0, N]$$

- goals when defining *f*
  - smoothness
  - predictable and local control
  - efficiency

parametric representation of splines

- splines: piecewise parametric polynomials
  - polynomials are smooth
  - controlled by small number of *local control points*
  - discontinuities at integer intervals

$$\mathbf{p}(t) = f(t) \qquad t \in [0, N]$$



splines - intuition

- define segment by “blending” *control points*

- join segments to form curve

defining splines

- pick segment interpolating function
- impose constraints to define segments
  - i.e. control points that define the spline
- impose constraints to join segments together

interpolating vs. approximating splines

- interpolating
  - pass through control points

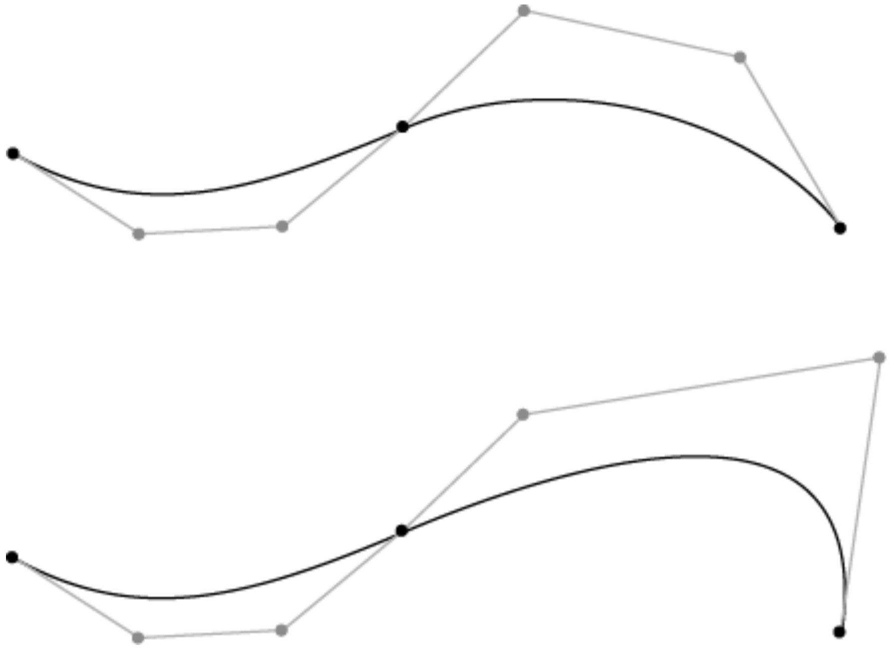
- approximating
  - guided by control points

smoothness

- smoothness described by degree of continuity
  - C^0: same position at each side of joints
  - C^1: same tangent at each side of joints
  - C^2: same curvature at each side of joints
  - C^n: *n*-th derivative defined at joints

control

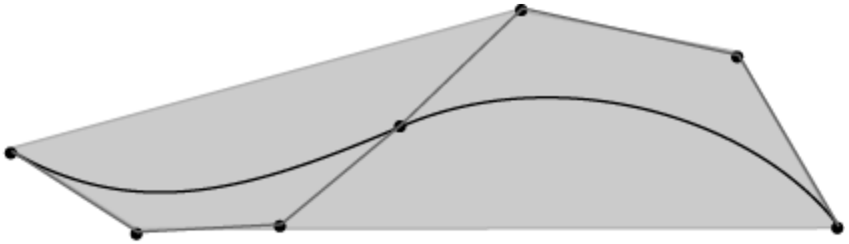
- local control
  - changing control points only affect locally the curve
    - easy to control
  - true for all splines



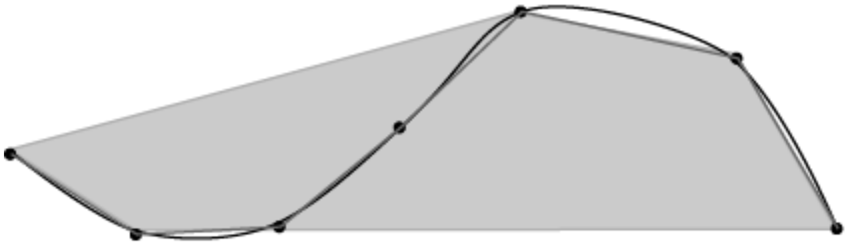
control

- convex hull property
  - convex hull: smallest convex region enclosing all points
    - predictable behavior
    - more efficient operations
  - only some splines

yes

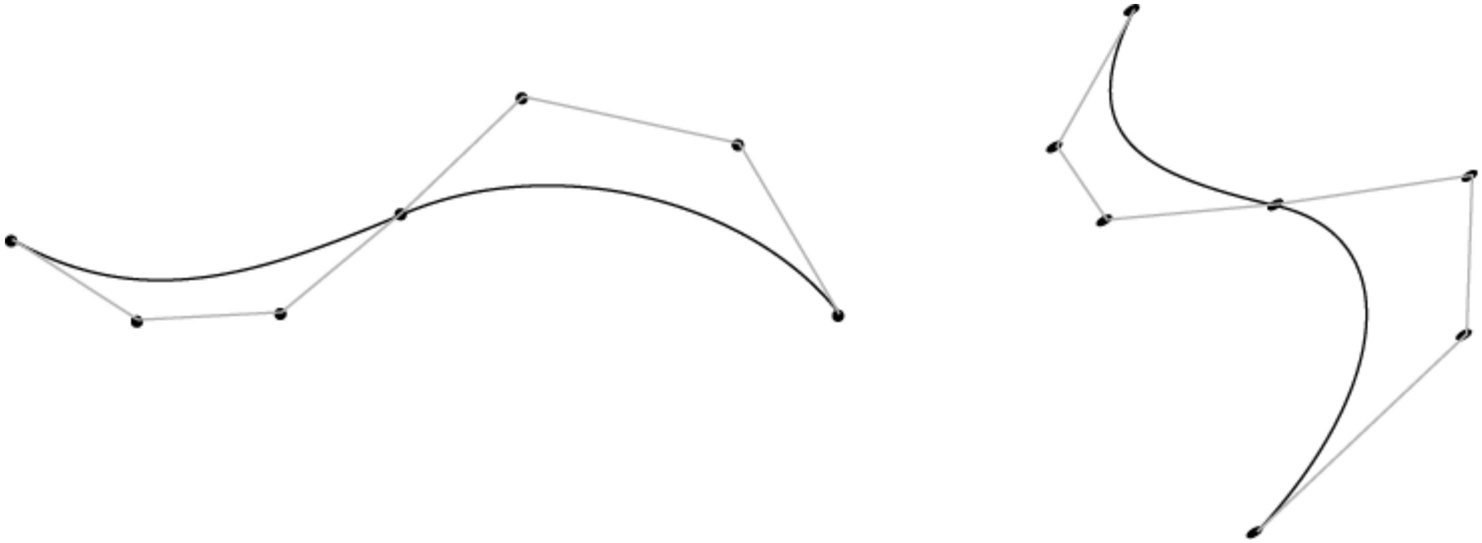


no



efficiency

- affine invariance
  - transforming the spline same as transforming controls
  - efficient algorithms, esp. combined with convex hull
  - true for all used splines



piecewise linear splines

- each segment is a linear function

$$\mathbf{p}(t) = t\mathbf{a} + \mathbf{b} \qquad t = [0,1]$$

- impose endpoint constraints

$$\begin{cases} \mathbf{p}(0) = \mathbf{p}_0 \\ \mathbf{p}(1) = \mathbf{p}_1 \end{cases} \Rightarrow \begin{cases} \mathbf{a} = \mathbf{p}_1 - \mathbf{p}_0 \\ \mathbf{b} = \mathbf{p}_0 \end{cases}$$

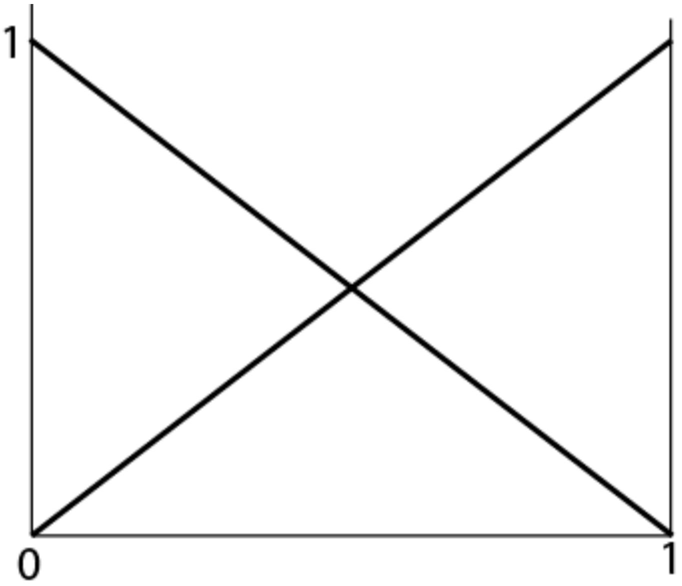
$$\mathbf{p}(t) = \mathbf{p}_0 + t(\mathbf{p}_1 - \mathbf{p}_0) \qquad t \in [0,1]$$

point blending interpretation

- can interpret as blending of points

$$\mathbf{p}(t) = (1-t)\mathbf{p}_0 + t\mathbf{p}_1 = b_0(t)\mathbf{p}_0 + b_1(t)\mathbf{p}_1 \qquad t \in [0,1)$$

- blending functions do not depend on points
  - different intervals only change control points



matrix notation

- write blending functions more conveniently

$$\mathbf{p}(t) = (1-t)\mathbf{p}_0 + t\mathbf{p}_1$$

$$\mathbf{p}(t) = \begin{bmatrix} t & 1 \end{bmatrix} \begin{bmatrix} -1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} \mathbf{p}_0 \\ \mathbf{p}_1 \end{bmatrix}$$

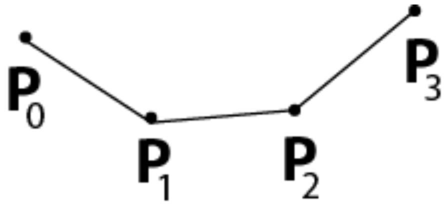
joining line segments

- impose C^0 continuity at joints
  - first segment

$$\mathbf{p}^0(t) \rightarrow \begin{cases} \mathbf{p}^0(0) = \mathbf{p}_0^0 \\ \mathbf{p}^0(1) = \mathbf{p}_1^0 \end{cases}$$

  - second segment

$$\mathbf{p}^1(t) \rightarrow \begin{cases} \mathbf{p}^1(0) = \mathbf{p}_0^1 \\ \mathbf{p}^1(1) = \mathbf{p}_1^1 \end{cases}$$



  - implies

$$\mathbf{p}^0(1) = \mathbf{p}^1(0) \rightarrow \mathbf{p}_1^0 = \mathbf{p}_0^1$$

- general formula
  - appropriately rename control points

$$\mathbf{p}(t) = b_0(t-k)\mathbf{p}_k + b_1(t-k)\mathbf{p}_{k+1} \qquad t \in [0, N], \ k = \text{floor}(t)$$

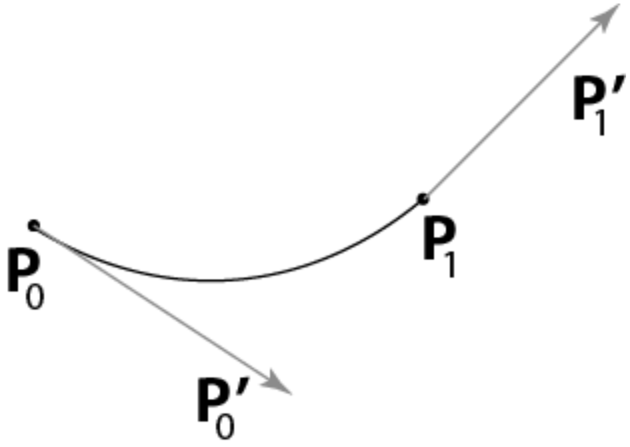
Hermite splines

- each segment is a cubic polynomial function

$$\mathbf{p}(t) = \mathbf{a}t^3 + \mathbf{b}t^2 + \mathbf{c}t + \mathbf{d}$$

- impose endpoints and tangents constraints

$$\begin{cases} \mathbf{p}(0) = \mathbf{p}_0 \\ \mathbf{p}(1) = \mathbf{p}_1 \\ \mathbf{p}'(0) = \mathbf{p}'_0 \\ \mathbf{p}'(1) = \mathbf{p}'_1 \end{cases}$$



Hermite splines

$$\mathbf{p}(t) = \mathbf{a}t^3 + \mathbf{b}t^2 + \mathbf{c}t + \mathbf{d}$$

$$\mathbf{p}'(t) = 3\mathbf{a}t^2 + 2\mathbf{b}t + \mathbf{c}$$

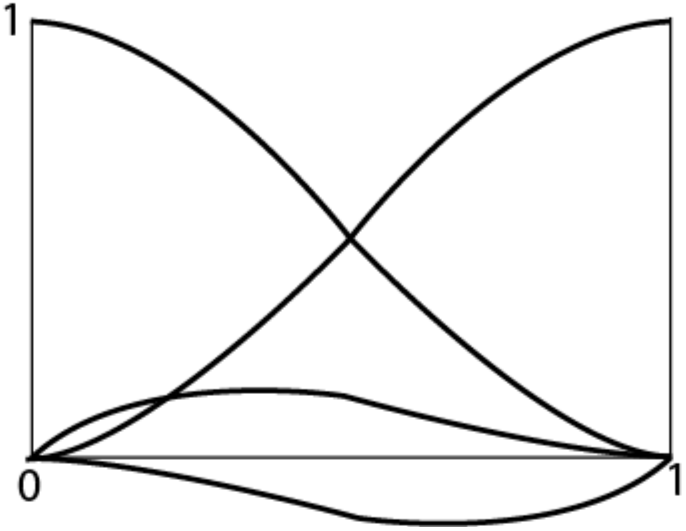
$$\begin{aligned} \mathbf{p}(0) &= \mathbf{d} \\ \mathbf{p}(1) &= \mathbf{a} + \mathbf{b} + \mathbf{c} + \mathbf{d} \\ \mathbf{p}'(0) &= \mathbf{c} \\ \mathbf{p}'(1) &= 3\mathbf{a} + 2\mathbf{b} + \mathbf{c} \end{aligned} \longrightarrow \begin{aligned} \mathbf{a} &= 2\mathbf{p}_0 - 2\mathbf{p}_1 + \mathbf{p}'_0 + \mathbf{p}'_1 \\ \mathbf{b} &= -3\mathbf{p}_0 + 3\mathbf{p}_1 - 2\mathbf{p}'_0 - \mathbf{p}'_1 \\ \mathbf{c} &= \mathbf{p}'_0 \\ \mathbf{d} &= \mathbf{p}_0 \end{aligned}$$

Hermite

- matrix formulation

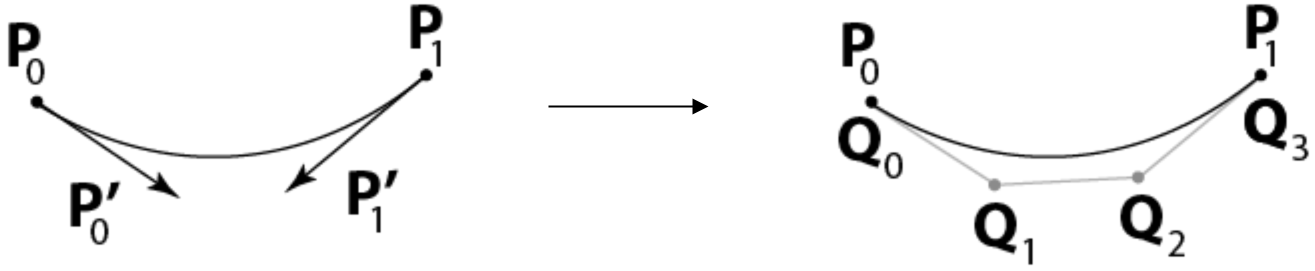
$$\mathbf{p}(t) = \begin{bmatrix} t^3 & t^2 & t & 1 \end{bmatrix} \begin{bmatrix} 2 & -2 & 1 & 2 \\ -3 & 3 & -2 & -1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \mathbf{p}_0 \\ \mathbf{p}_1 \\ \mathbf{p}'_0 \\ \mathbf{p}'_1 \end{bmatrix}$$

- blending functions

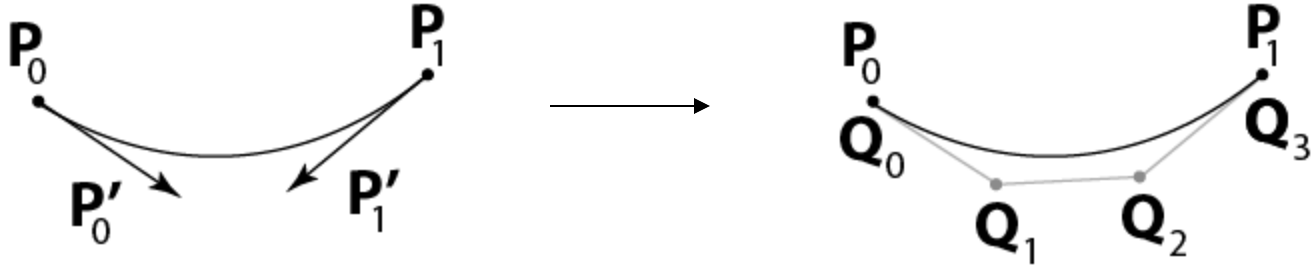


Bezier splines

- Hermite splines has points and vectors controls
  - would like to use just points
  - insight: specify tangents as difference of points
    - choose appropriate scaling value, see later



Bezier splines



$$\mathbf{p}_0 = \mathbf{q}_0$$

$$\mathbf{p}_1 = \mathbf{q}_3$$

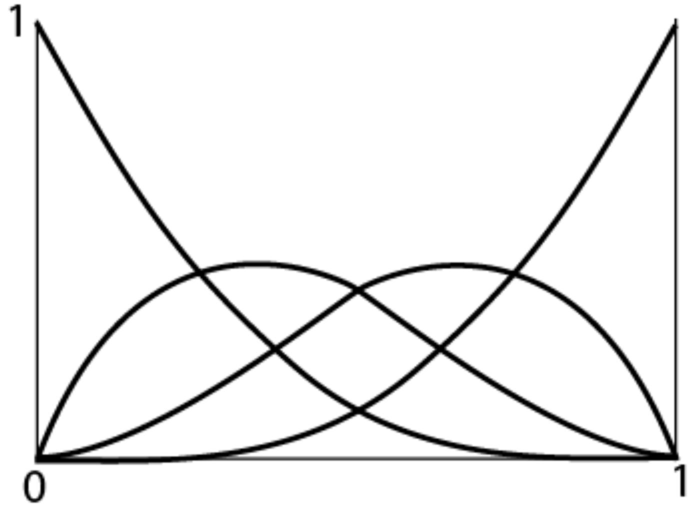
$$\mathbf{p}'_0 = 3(\mathbf{q}_1 - \mathbf{q}_0)$$

$$\mathbf{p}'_1 = 3(\mathbf{q}_3 - \mathbf{q}_2)$$

$$\mathbf{p}(t) = \begin{bmatrix} t^3 & t^2 & t & 1 \end{bmatrix} \begin{bmatrix} -1 & 3 & -3 & 1 \\ 3 & -6 & 3 & 0 \\ -3 & 3 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \mathbf{q}_0 \\ \mathbf{q}_1 \\ \mathbf{q}_2 \\ \mathbf{q}_3 \end{bmatrix}$$

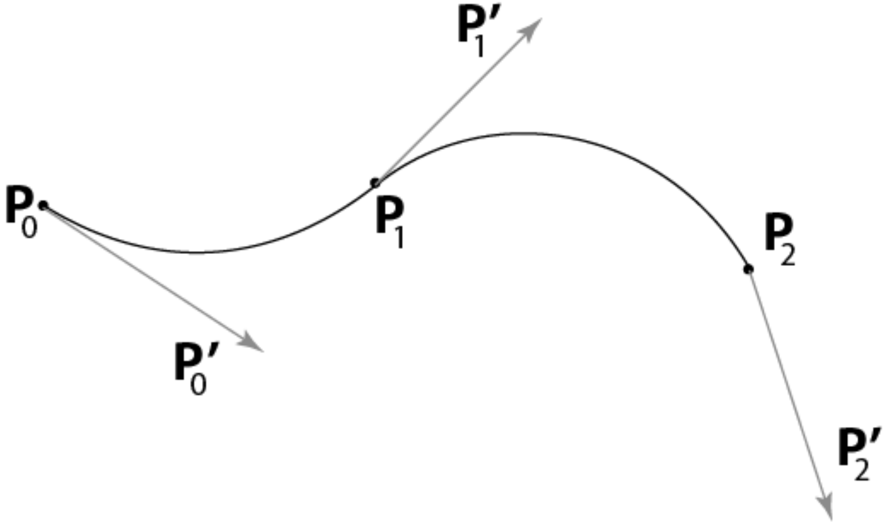
Bezier splines

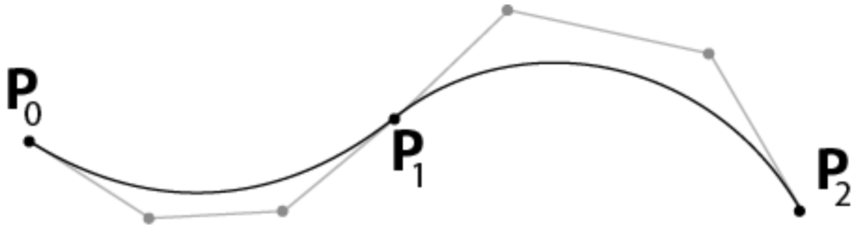
- blending functions



piecewise cubic splines – smoothness

- C^1 at joints by imposing equal tangents
  - Hermite: same tangents
  - Bezier: collinear control points
    - geometric continuity if length of tangent differs





piecewise cubic splines – control

- local control
  - comes from the formulation by segments
  - for each segment, curve defined by 4 control points

- convex hull
  - when blending positions

$$b_i(t) \geq 0 \quad \text{and} \quad \sum_{i=0}^{3} b_i(t) = 1$$

piecewise cubic splines – affine invariance

- affine invariance
  - affine is combination of linear and translation
  - blending functions sum to 1

$$X(\mathbf{p}(t)) = M\mathbf{p}(t) + \mathbf{t} = M\left(\sum_{i=0}^{3} b_i(t)\mathbf{p}_i\right) + \mathbf{t} =$$

$$= \sum_{i=0}^{3} b_i(t) M\mathbf{p}_i + \sum_{i=0}^{3} b_i(t)\mathbf{t} =$$

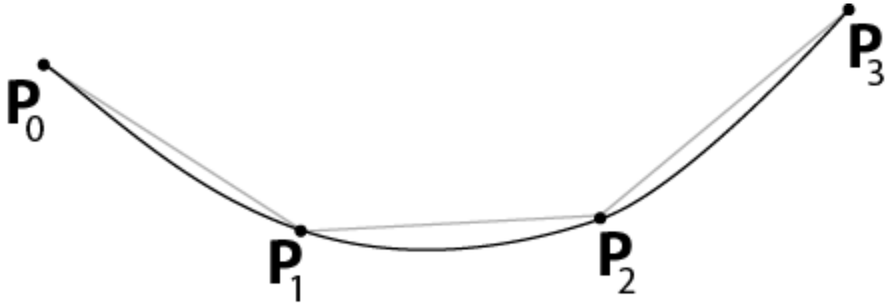
$$= \sum_{i=0}^{3} b_i(t)(M\mathbf{p}_i + \mathbf{t}) = \sum_{i=0}^{3} b_i(t) X(\mathbf{p}_i)$$

Bezier splines

- widely used, especially in 2D
  - primitive in PDF
- represent C^1 and C^0 curves with corners
- easily add point at any position

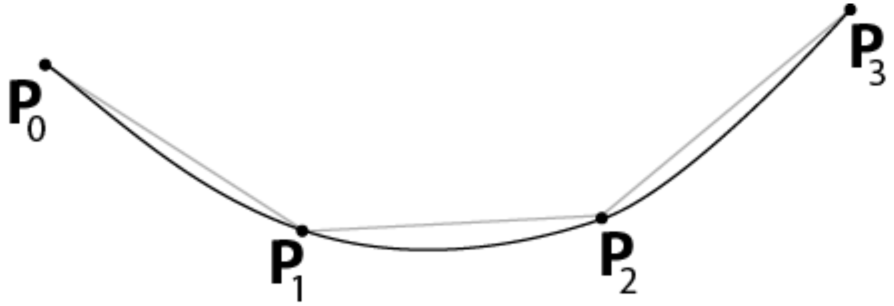
Catmull-Rom splines

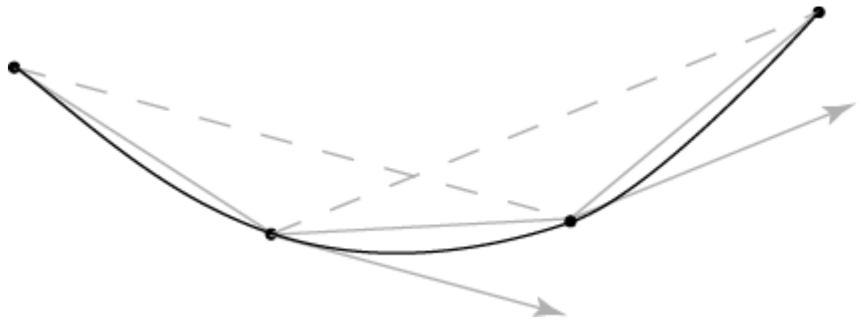
- interpolating spline
  - no convex hull property
- as Hermite, derivatives automatically determined
  - using adjacent control points
  - end tangent using either adding point or zers



Catmull-Rom splines

$$\mathbf{p}'_k = \left(\mathbf{p}_{k+1} - \mathbf{p}_{k-1}\right)/\,2$$



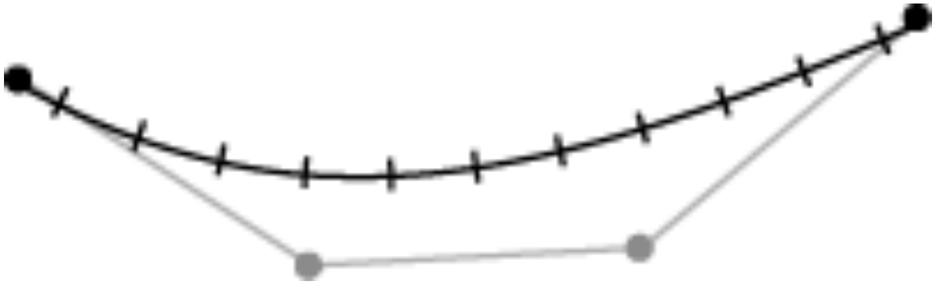


drawing splines

- approximate with a sequence of line segments
  - efficiency: fast evaluation, small number of segments
  - guarantees on accuracy
- approaches
  - uniform subdivision in *t* (fast)
  - recursive subdivision (small number of segments)

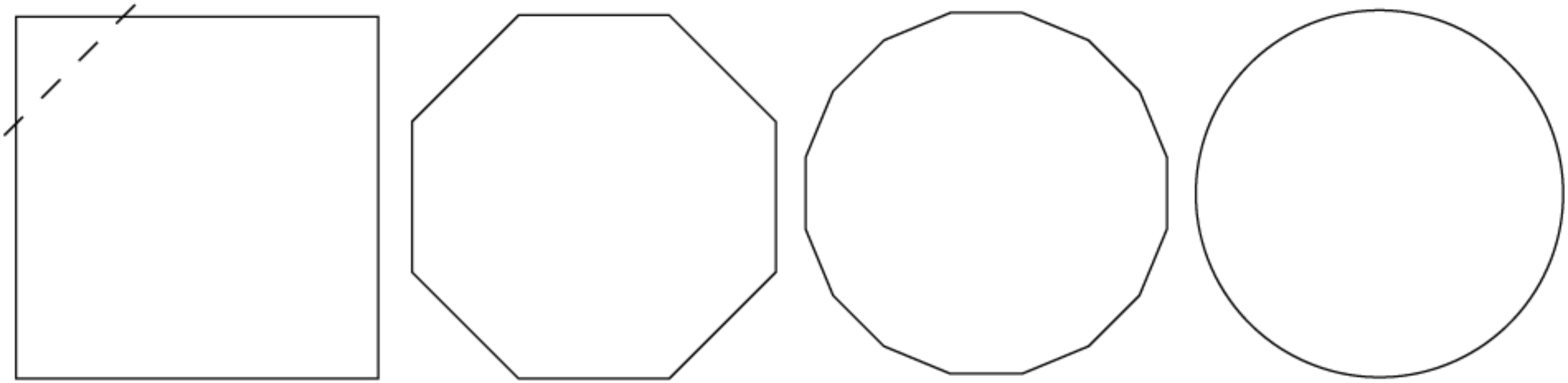
uniform subdivision

- evaluate spline at fixed *t* intervals
  - can be done efficiently



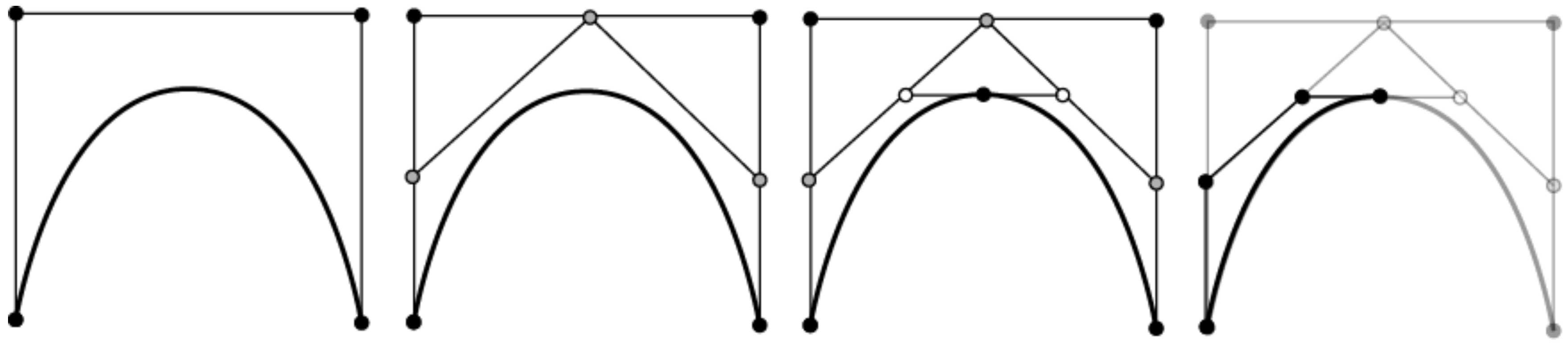
adaptive subdivision - Bezier

- recursively subdivide spline
- until line segments approximate well curve



De Casteljau algorithm - Bezier

- recursively do
  - connect midpoints of the control polygons
  - connect midpoints of the new segments
  - the midpoint of this last segment is on the curve
  - and splits the curve in two Bezier segments
- stop when control polygon is close to collinear



B-Splines

- would like C^2 continuity at joints
  - give up interpolation
- impose 3 continuity constraints at joints

$$\mathbf{p}(t) = \begin{bmatrix} t^3 & t^2 & t & 1 \end{bmatrix} \frac{1}{6} \begin{bmatrix} -1 & 3 & -3 & 1 \\ 3 & -6 & 3 & 0 \\ -3 & 0 & 3 & 0 \\ 1 & 4 & 1 & 0 \end{bmatrix} \begin{bmatrix} \mathbf{q}_{k-1} \\ \mathbf{q}_k \\ \mathbf{q}_{k+1} \\ \mathbf{q}_{k+2} \end{bmatrix}$$

other splines

- many other types

- non-uniform B-splines
  - discontinuities not evenly spaces
- non-uniform rational B-splines (NURBS)
  - ratios of non-uniform B-splines
  - invariance under perspective
  - can represent conic sections exactly
  - often used in 3D

spline equivalence

- all splines seen so far are equivalent
  - represented by 4x4 matrices

- can convert control points from one to other
  - algorithms can be based on the most efficient
  - UIs can be based on the most user-friendly

2D vs. 3D splines

- often use 2D splines in 3D
  - by projecting onto a plane
- 3D parametric splines have same formulation
  - just use 3D vectors vs. 2D ones